The Open University

MS221
Exploring Mathematics

Chapter C3

Taylor polynomials

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The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1997. Second edition 2003. Reprinted with corrections 2005. Reprinted 2006.

Edited, designed and typeset by The Open University, using the Open University TEX System.

Printed and bound in the United Kingdom by The Charlesworth Group, Wakefield.

ISBN 0 7492 5561 7

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Contents

Study guide

There are five sections in this chapter. They are intended to be studied consecutively in five study sessions. Section 5 requires the use of the computer and Computer Book C.

The pattern of study for each session might be as follows.

Study session 1: Section 1.

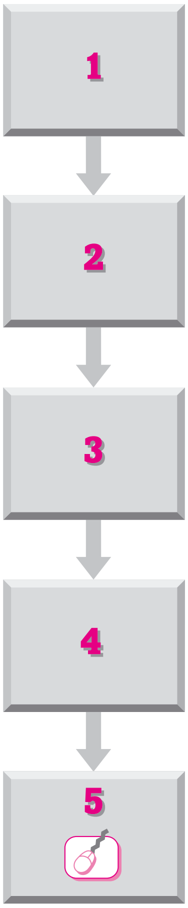
Study session 2: Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Study session 5: Section 5.

Each session requires two to three hours. Sections 2 and 4 may take longer to study than the other sections. Subsection 2.4 will not be assessed.



The optional Video Band C(v) 'Algebra Workout – Taylor Polynomials' could be viewed at any stage during your study of this chapter.

Introduction

Many applications of mathematics require the evaluation or approximate evaluation of real functions at particular points in their domains. In the case of *polynomial functions*, often abbreviated to *polynomials*, this is relatively straightforward, since it is just a matter of performing simple arithmetic. For example, if the function f has the rule

Polynomial functions were introduced in MST121 Chapter A3. They have the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $n \geq 0$. If $a_n \neq 0$, then the polynomial has degree n.

$$f(x) = 6 - 3x + 2x^2 + x^3,$$

then the value of f at 3 is

$$\begin{aligned} f(3) &= 6 - 3 \times 3 + 2 \times 3^2 + 3^3 \\ &= 6 - 9 + 18 + 27 = 42. \end{aligned}$$

By contrast, many of the functions common in mathematics, science and technology, such as $\sin x$, e^x and $\ln x$, cannot be easily evaluated at given points in their domains except in a few special cases.

You may have wondered how a calculator or mathematical software package ascribes an approximate numerical value to $\ln 3$, for example. There are various ways in which this can be done, but one method used by many software packages involves approximating the natural logarithm function by a polynomial function. This allows $\ln 3$ to be evaluated to the accuracy of the computer using just the simple arithmetical operations of addition, subtraction and multiplication.

In this chapter you will study a particular way of approximating functions by polynomials, called *Taylor polynomials*. By using suitable Taylor polynomials, we can approximate many functions to any required accuracy. In fact, software packages do not use Taylor polynomials to approximate functions, since more efficient (but more complicated) polynomial methods exist, but by studying Taylor polynomials you will learn about the basic ideas of polynomial approximation.

Polynomial approximations are also important because it is straightforward to multiply polynomials together, and to differentiate and integrate them. Furthermore, polynomial approximations allow complex problems to be described by simple mathematical models, making these problems easier both to understand and to solve.

Taylor polynomials are also of theoretical importance, as they lead naturally to a way of representing functions by infinite series, at least for some values in their domains. Such representations are called *Taylor series*.

In Section 1 you will study the approximation of functions by linear and quadratic functions. This is extended to approximation by Taylor polynomials of higher degree in Section 2 and then to Taylor series in Section 3. In Section 4 various methods are given for using known Taylor series to derive Taylor series for further functions. Finally, in Section 5, you will explore Taylor polynomials and Taylor series using your computer.

1 Linear and quadratic Taylor polynomials

1.1 Linear Taylor polynomials

We begin our study of polynomial approximations by exploring the approximation of functions by linear functions. Linear functions have the form $p(x) = a_0 + a_1x$, where a_0 and a_1 are constants.

In most cases a function cannot be approximated by a linear function across the whole of its domain. What we are interested in here is the approximation of a given function f by a linear function close to some point a in the domain of f. For example, Figure 1.1 shows part of the graph of a smooth function f. Suppose that we want to find a linear function

$$p(x) = a_0 + a_1x$$

Be careful not to confuse the point a in the domain of f with the coefficients a_0 and a_1 of the linear function.

that approximates f for values of x close to a. Geometrically, this means that we seek a straight line, with equation $y = a_0 + a_1x$, that approximates the curve $y = f(x)$ close to the point $(a, f(a))$.

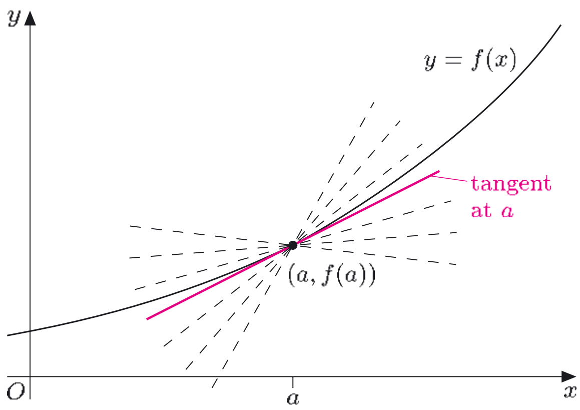


Figure 1.1 A smooth function f and lines through the point $(a, f(a))$

It seems appropriate to choose a line that passes through the point $(a, f(a))$. The most suitable such line appears to be the *tangent* to the graph of f at a; this is the line whose gradient is the same as that of the graph of f at $(a, f(a))$. Figure 1.1 shows several lines passing through $(a, f(a))$, and you can see that the tangent appears to be the best approximation for f close to $(a, f(a))$, in the sense that any other line through this point 'moves away more quickly' from the graph of f. The linear function whose graph is this tangent is called the **linear Taylor polynomial** about a for f. For any point x close to a, the value of this polynomial at x is an approximation for $f(x)$.

In some texts linear Taylor polynomials are called *tangent approximations*.

Example 1.1 Approximating the exponential function

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = e^x$.

(b) Use this polynomial to find an approximation for $e^{0.1}$.

Solution

(a) The linear Taylor polynomial about 0 for f is the function whose graph is the line that passes through the point $(0, f(0))$ and has gradient equal to that of the curve $y = f(x)$ at $(0, f(0))$.

In this case $(0, f(0)) = (0, e^0) = (0, 1)$. Also $f'(x) = e^x$, so the gradient of the curve at $(0, 1)$ is $f'(0) = e^0 = 1$. Thus the required line passes through the point $(0, 1)$ and has gradient 1.

Since its gradient is 1, the line has an equation of the form $y = a_0 + x$. Since it passes through $(0, 1)$, we have $1 = a_0 + 0$; that is, $a_0 = 1$. Thus the line has equation $y = 1 + x$, and so the linear Taylor polynomial is $p(x) = 1 + x$.

A line with gradient a_1 has an equation of the form $y = a_0 + a_1 x$.

(b) The approximation for $e^{0.1}$ given by the polynomial p is

$$p(0.1) = 1 + 0.1 = 1.1.$$

The value of $e^{0.1}$ is 1.105 171, to six decimal places.

The graphs of the function $f(x) = e^x$ and the linear polynomial $p(x) = 1 + x$ that was found in Example 1.1 are illustrated in Figure 1.2. You can see that, as expected, the linear approximation appears to be good for values of x close to zero, but its accuracy decreases as the distance of x from 0 increases.

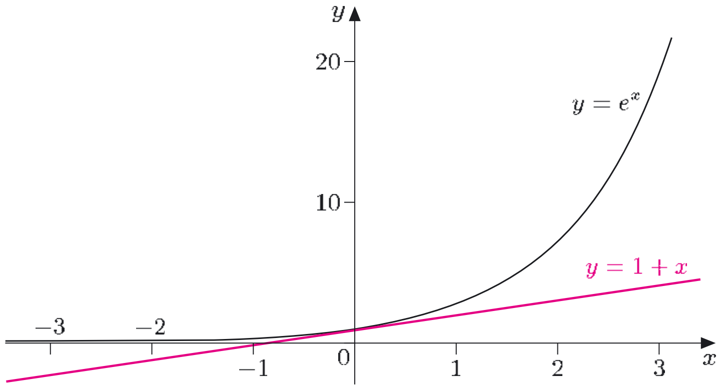


Figure 1.2 Linear Taylor polynomial about 0 for $f(x) = e^x$

To help us gauge the accuracy of a polynomial approximation to a function, it is useful to introduce the idea of a **remainder function**. If f is a function, and p is a polynomial that is intended to approximate f, then we define the remainder function r associated with f and p by

$$r(x) = f(x) - p(x),$$

for all x in the domain of f. For a given value of x, the closer the value of $r(x)$ is to zero, the better $p(x)$ is as an approximation for $f(x)$. A value of a remainder function is often referred to simply as a remainder.

The remainder function gives the ‘error in the approximation’.

Table 1.1 overleaf lists, to four decimal places, some values of the function $f(x) = e^x$, together with corresponding values for the linear Taylor polynomial $p(x) = 1 + x$ about 0 and the associated remainder function $r(x) = e^x - (1 + x)$. The small values of $r(x)$ for values of x close to 0 indicate that $1 + x$ is a fairly good approximation for $f(x) = e^x$ near $x = 0$. It seems that the closer the value of x to 0, the smaller the magnitude of $r(x)$. The remainder at $x = 0$ is zero because, by design, $p(0)$ is equal to $f(0)$. All the remainders for $x \neq 0$ are positive because the graph of p lies beneath the graph of f for both positive and negative values of x.

For example, for the functions f and p in Example 1.1, the remainder at $x = 0.1$ is

$$\begin{aligned} r(0.1) &= e^{0.1} - p(0.1) \\ &= 0.005\,171, \end{aligned}$$

to six decimal places.

Table 1.1

x	$f(x) = e^x$	$p(x) = 1 + x$	$r(x) = f(x) - p(x)$
-1	0.3679	0	0.3679
-0.75	0.4724	0.25	0.2224
-0.5	0.6065	0.5	0.1065
-0.25	0.7788	0.75	0.0288
0	1	1	0
0.25	1.2840	1.25	0.0340
0.5	1.6487	1.5	0.1487
0.75	2.1170	1.75	0.3670
1	2.7183	2	0.7183

The values of e^x were found using a calculator.

In the first activity you are asked to find a linear Taylor polynomial for the sine function, and investigate its accuracy.

Activity 1.1 A linear Taylor polynomial for sin

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \sin x$.

(b) Investigate the accuracy of this approximation by using your calculator to compile a table similar to Table 1.1. Comment on what you notice about the values of the remainder function.

Remember that your calculator should be in radian mode.

Solutions are given on page 50.

Notice that the linear Taylor polynomial about 0 for the sine function contains no constant term; it is $p(x) = x$. This happens because the graph of the sine function passes through the origin. The graphs of $f(x) = \sin x$ and $p(x) = x$ are shown in Figure 1.3. The graph of p lies below that of f for $x < 0$ and above it for $x > 0$, confirming the signs of the remainders calculated in Activity 1.1(b).

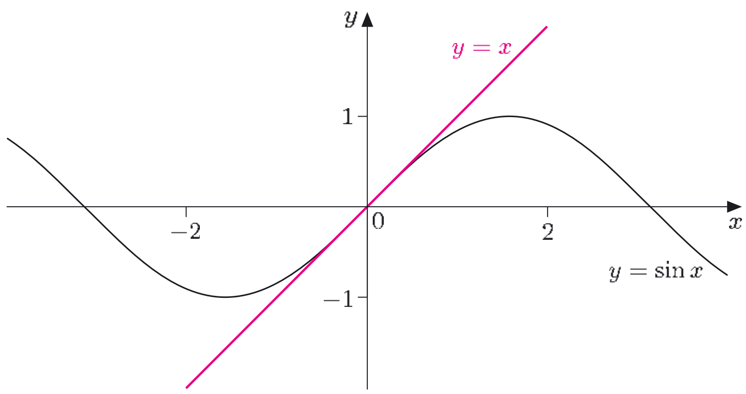


Figure 1.3 Linear Taylor polynomial about 0 for $f(x) = \sin x$

In Activity 1.1 you saw that the linear function $p(x) = x$ is an approximation for the function $f(x) = \sin x$ for values of x close to 0. This approximation is consistent with the formula

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1,$$

obtained earlier.

See Chapter C1, Subsection 1.2.

Activity 1.2 A linear Taylor polynomial for cos

(a) Find the linear Taylor polynomial about 0 for the function $f(x) = \cos x$.

(b) Use this polynomial to find an approximation for $\cos(0.2)$, and use your calculator to find the value of the associated remainder to six decimal places.

Solutions are given on page 50.

Notice that the linear Taylor polynomial about 0 for the cosine function contains no term in x and is therefore a constant function; it is $p(x) = 1$. This happens because the graph of the function $f(x) = \cos x$ has gradient zero at the point where $x = 0$. The graphs of $f(x) = \cos x$ and $p(x) = 1$ are shown in Figure 1.4.

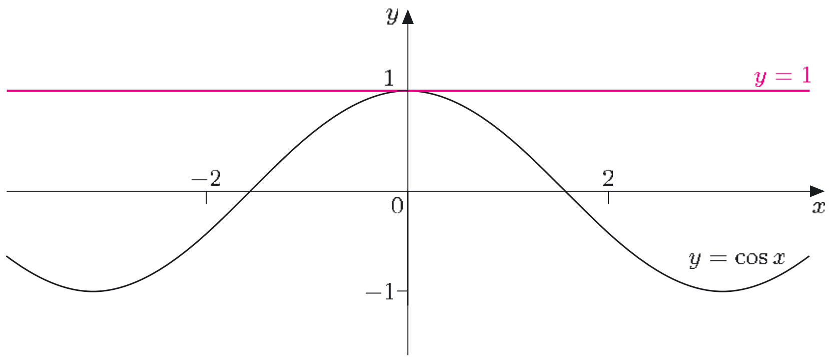


Figure 1.4 Linear Taylor polynomial about 0 for $f(x) = \cos x$

All the linear Taylor polynomials that you have seen so far in this subsection have been about 0. In the next activity you are asked to find linear Taylor polynomials about another point.

Activity 1.3 More linear Taylor polynomials

Find the linear Taylor polynomial about 1 for each of the following functions.

(a) $f(x) = \ln x - 1/x$

(b) $f(x) = e^x$

Solutions are given on page 50.

In each case the graph of the required polynomial is the line which passes through the point $(1, f(1))$ and has gradient equal to $f'(1)$ at this point.

In part (b) of Activity 1.3 you obtained the linear Taylor polynomial about 1 for the function $f(x) = e^x$, while in Example 1.1 the linear Taylor polynomial about 0 was obtained for the same function. The graphs of these linear Taylor polynomials are shown in Figure 1.5. As expected, it appears that one of these polynomials approximates e^x for values of x close to 1, while the other approximates e^x for values of x close to 0. This illustrates that, in general, polynomial approximations about different points are different polynomials.

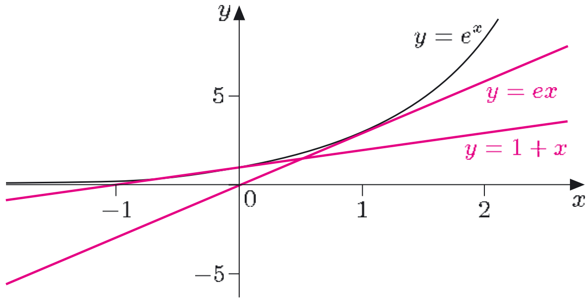


Figure 1.5 Linear Taylor polynomials about 0 and 1 for $f(x) = e^x$

In the next activity you will use a linear Taylor polynomial to approximate a square root.

Activity 1.4 A further linear Taylor polynomial

(a) Show that the linear Taylor polynomial about 0 for the function

$$f(x) = (1 + x)^k, \quad \text{where } k > 0,$$

is

$$p(x) = 1 + kx.$$

(b) Use the polynomial p from part (a) with $k = \frac{1}{2}$ and $x = 0.01$ to find an approximate value for $\sqrt{1.01}$. Use your calculator to find, to six decimal places, the value of the associated remainder.

Solutions are given on page 50.

1.2 Quadratic Taylor polynomials about 0

We now look at approximating functions by quadratic functions. You might expect this to give greater accuracy than approximating a function by a linear function, and this is usually the case. For simplicity, in this subsection we consider approximations about 0 only. However, in Section 2 we study approximations by polynomials of any degree n, and there you will see how to find polynomial approximations about a general point a.

Suppose that f is a differentiable function whose domain contains 0. In Subsection 1.1 you saw how to approximate f close to 0 by a function p of the form $p(x) = a_0 + a_1x$, which was chosen to be the function whose graph is the tangent to the graph of f at 0. In other words, p was chosen to be the linear function that satisfies the following two conditions:

Recall from Chapter C1, Subsection 1.1 that 'differentiable function' means the same as 'smooth function'.

(i) the values of the function and the approximation are equal at 0; that is, $p(0) = f(0)$;

(ii) the values of the first derivatives of the function and the approximation are equal at 0; that is, $p'(0) = f'(0)$.

Usually this linear approximation is good for values of x close to zero, but its accuracy decreases as x moves away from zero.

Suppose that we now wish instead to consider approximating f close to 0 by a function p of the form

$$p(x) = a_0 + a_1x + a_2x^2.$$

It seems sensible to require this new function p to satisfy conditions (i) and (ii) above. As p now has three coefficients, we can also impose a third condition, and a natural one to choose is:

(iii) the values of the second derivatives of the function and the approximation are equal at 0; that is, $p''(0) = f''(0)$.

We can impose this condition provided that $f''(0)$ exists; that is, provided that f is twice-differentiable at 0.

Many real functions can be differentiated as often as we wish at all points in their domains. These include polynomial, rational, trigonometric, exponential and logarithmic functions, and combinations and compositions of these.

You have seen that conditions (i) and (ii) ensure that the graphs of the function and the approximation both pass through the same point $(0, f(0))$ and have the same gradient at that point. Condition (iii) ensures that the function and the approximation also have the same rate of change of gradient at 0. (Roughly speaking, this means that their graphs have the same 'curvature' at the point $(0, f(0))$.)

The polynomial p that satisfies conditions (i), (ii) and (iii) is called the **quadratic Taylor polynomial** about 0 for f. For any point x close to 0, the value of $p(x)$ is an approximation for $f(x)$.

In some cases, the polynomial $p(x) = a_0 + a_1x + a_2x^2$ that satisfies conditions (i), (ii) and (iii) has $a_2 = 0$ and so is *not* a quadratic polynomial, but has degree 1 or less. If this happens, then we still refer to the approximating polynomial as the *quadratic* Taylor polynomial about 0 for f. This means that a quadratic Taylor polynomial is not necessarily a quadratic polynomial! You will see an example of this later in this subsection.

Example 1.2 A quadratic Taylor polynomial for exp

Find the quadratic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

Let the polynomial that we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

We need to determine what the values of the constants a_0, a_1 and a_2 must be to ensure that the value of p at 0, and the first and second derivative values of p at 0, are the same as those of f.

First we ensure that $p(0) = f(0)$. We have

$$f(x) = e^x \qquad \text{and} \quad p(x) = a_0 + a_1 x + a_2 x^2,$$

so

$$f(0) = e^0 = 1 \quad \text{and} \quad p(0) = a_0.$$

Thus we must have $a_0 = 1$.

Next we ensure that $p'(0) = f'(0)$. We have

$$f'(x) = e^x \qquad \text{and} \quad p'(x) = a_1 + 2a_2 x,$$

so

$$f'(0) = e^0 = 1 \quad \text{and} \quad p'(0) = a_1.$$

Thus we must have $a_1 = 1$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$f''(x) = e^x \qquad \text{and} \quad p''(x) = 2a_2,$$

so

$$f''(0) = e^0 = 1 \quad \text{and} \quad p''(0) = 2a_2.$$

Thus we must have $2a_2 = 1$; that is, $a_2 = \frac{1}{2}$.

Hence the quadratic Taylor polynomial about 0 for $f(x) = e^x$ is

$$p(x) = 1 + x + \tfrac{1}{2}x^2.$$

The graphs of $f(x) = e^x$ and the approximation $p(x) = 1 + x + \frac{1}{2}x^2$ that was found in Example 1.2 are shown in Figure 1.6. The quadratic function p appears to be a more accurate approximation for $f(x) = e^x$ for values of x close to 0 than the linear function that we found earlier; see Figure 1.2, page 7. More evidence for this is provided by Table 1.2, opposite.

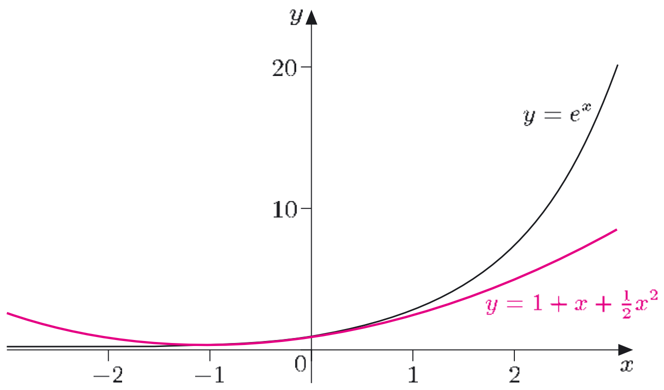


Figure 1.6 Quadratic Taylor polynomial about 0 for $f(x) = e^x$

Table 1.2 lists, to four decimal places, some values of the function $f(x) = e^x$, together with corresponding values for the quadratic Taylor polynomial $p(x) = 1 + x + \frac{1}{2}x^2$ and the associated remainder function $r(x) = f(x) - p(x)$. If you compare this table with Table 1.1 on page 8, then you will see that, for each of the listed non-zero values of x, the quadratic approximation is more accurate than the linear approximation (since in each case the remainder has smaller magnitude). Also, as for the linear approximation, it appears that the closer the value of x is to 0, the more accurate is the quadratic approximation. The remainders for $x < 0$ are negative because the graph of the approximation p lies above the graph of the function f for such values of x, whereas those for $x > 0$ are positive because the graph of the approximation p lies below the graph of the function f for such values of x.

Table 1.2

x	$f(x) = e^x$	$p(x) = 1 + x + \frac{1}{2}x^2$	$r(x) = f(x) - p(x)$
−1	0.3679	0.5	−0.1321
−0.75	0.4724	0.5313	−0.0589
−0.5	0.6065	0.625	−0.0185
−0.25	0.7788	0.7813	−0.0024
0	1	1	0
0.25	1.2840	1.2813	0.0028
0.5	1.6487	1.625	0.0237
0.75	2.1170	2.0313	0.0858
1	2.7183	2.5	0.2183

In the next two activities you are asked to find quadratic Taylor polynomials for the cosine and sine functions.

Activity 1.5 A quadratic Taylor polynomial for cos

(a) Find the quadratic Taylor polynomial about 0 for the function

$$f(x) = \cos x.$$

(b) Use this polynomial to find an approximation for $\cos(0.2)$, and use your calculator to find the value of the associated remainder to six decimal places. Compare this approximation for $\cos(0.2)$ with the one found in Activity 1.2. Which is better?

Solutions are given on page 51.

The graphs of $f(x) = \cos x$ and the approximation $p(x) = 1 - \frac{1}{2}x^2$ found in Activity 1.5 are shown in Figure 1.7.

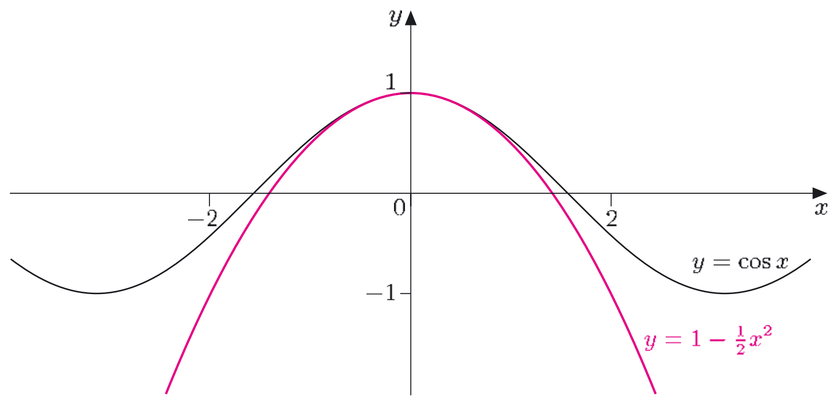


Figure 1.7 Quadratic Taylor polynomial about 0 for $f(x) = \cos x$

Activity 1.6 A quadratic Taylor polynomial for sin

Find the quadratic Taylor polynomial about 0 for the function

$$f(x) = \sin x.$$

A solution is given on page 51.

The quadratic Taylor polynomial about 0 for the sine function is an example of a quadratic Taylor polynomial that is not a quadratic polynomial. The quadratic Taylor polynomial about 0 for the sine function is the same as the linear Taylor polynomial about 0 for the sine function, which is shown in Figure 1.3 on page 8.

We have now found both linear and quadratic Taylor polynomials about 0 for each of the functions exp, sin and cos. These are listed in Table 1.3.

Table 1.3

Function	Linear Taylor polynomial about 0	Quadratic Taylor polynomial about 0
e^x	$1 + x$	$1 + x + \frac{1}{2}x^2$
$\sin x$	x	x
$\cos x$	1	$1 - \frac{1}{2}x^2$

Notice that for each of these three functions the constant term and the term in x are the same in the linear Taylor polynomial as in the quadratic Taylor polynomial. Hence in each case the quadratic Taylor polynomial can be obtained from the linear Taylor polynomial by adding the appropriate term in x^2 (the term is $0x^2$ in the case of sin). You will see in Section 2 that this property holds for every function f for which these approximations can be found, and that it extends to higher degree Taylor polynomials.

Summary of Section 1

This section has introduced:

- ◇ the linear Taylor polynomial for a function f about any point a in its domain;
- ◇ the fact that the linear Taylor polynomial about a for a function f approximates f for values of x close to a, and that usually the accuracy of the approximation decreases the further x is from a;
- ◇ the quadratic Taylor polynomial about 0 for a function f;
- ◇ the fact that the quadratic Taylor polynomial about 0 for a function f approximates f for values of x close to 0, and that usually the accuracy of the approximation decreases the further x is from 0;
- ◇ the fact that the quadratic Taylor polynomial about 0 for a function f usually provides greater accuracy than the corresponding linear Taylor polynomial.

Exercises for Section 1

Exercise 1.1

Find the linear Taylor polynomial about 1 for the function $f(x) = 1/x^2$.

Exercise 1.2

Find the linear Taylor polynomial about 0 for the function $f(x) = (8 + x)^{1/3}$. Hence find an approximate value for $\sqrt[3]{8.01}$, and use your calculator to find the associated remainder to six decimal places.

Exercise 1.3

Find the quadratic Taylor polynomial about 0 for the function $f(x) = \tan x$. Hence find an approximate value for $\tan(-0.1)$, and use your calculator to find the associated remainder to six decimal places.

2 Taylor polynomials

In this section we look at approximating functions by polynomials of any degree. We begin in Subsection 2.1 by considering approximations about 0 only. Later, in Subsection 2.2, we generalise the discussion to polynomial approximations about any point in the domain of a function. In Subsection 2.3 we use polynomials to find approximate values for functions. The final subsection, Subsection 2.4, explores the mathematics underlying Subsection 2.3.

2.1 Taylor polynomials of degree n about 0

Here 'suitable' means that f must be differentiable at 0 the required number of times.

From what you saw in Section 1, you might guess that for any suitable function f whose domain contains 0, we can obtain more and more accurate approximations for f close to 0 by taking polynomials of higher and higher degree, and choosing the coefficients to ensure that the values of higher and higher derivatives of the polynomial at 0 are the same as those of the corresponding derivatives of f.

For example, to try to improve on the approximation provided by the quadratic Taylor polynomial about 0, we could attempt to approximate f by a function of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ whose value is the same as that of f at 0, and whose first, second and third derivatives have the same values at 0 as the corresponding derivatives of f.

In fact, for any value of n, it is possible to find a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

whose value is the same as that of f at 0, and whose first, second, third, $\ldots$, nth derivatives have the same values at 0 as the corresponding derivatives of f. It is usually true that the larger the value of n, the better $p(x)$ is as an approximation for $f(x)$, for values of x close to 0.

In Example 1.2 we found the quadratic Taylor polynomial about 0 for the exponential function. You may find it helpful to review that example.

Recall that $f^{(n)}(x)$ denotes the nth derivative of f at x. The third derivative $f^{(3)}(x)$ is sometimes written as $f'''(x)$.

The polynomial $p(x)$ can be found by using the method of Example 1.2, but continuing with derivatives up to the nth, rather than just the second. However, it would be time-consuming to work through this method each time we wished to find the polynomial $p(x)$ for a particular suitable function f and value of n. To avoid having to do so, we shall now work through the method for a general function f and general value of n. Thus $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, $\ldots$, $f^{(n)}(0)$ will not be evaluated, but will appear in this form throughout the working. The result will be a general formula for $p(x)$ that will be available for us to apply in particular cases.

We need to determine what the values of the constants $a_0, a_1, a_2, \ldots, a_n$ in the polynomial $p(x)$ must be to ensure that the value of p at 0, and the first, second, third, $\ldots$, nth derivative values of p at 0, are the same as those of f.

First we ensure that $p(0) = f(0)$. We have

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n,$$

so

$$p(0) = a_0.$$

Thus we must have $a_0 = f(0)$.

Next we ensure that $p'(0) = f'(0)$. We have

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1},$$

so

$$p'(0) = a_1.$$

Thus we must have $a_1 = f'(0)$.

Then we ensure that $p''(0) = f''(0)$. We have

$$p''(x) = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + \cdots + n(n-1)a_nx^{n-2},$$

so

$$p''(0) = 2a_2.$$

Thus we must have $2a_2 = f''(0)$; that is, $a_2 = \frac{1}{2!}f''(0)$.

In these calculations the number 2 has been written as 2!, and the product 3×2 has been simplified to 3! rather than 6, to highlight the emerging pattern. Recall that in general $k! = k(k-1) \times \cdots \times 2 \times 1$.

Then we ensure that $p^{(3)}(0) = f^{(3)}(0)$. We have

$$p^{(3)}(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3},$$

so

$$p^{(3)}(0) = 3! \times a_3.$$

Thus we must have $3! \times a_3 = f^{(3)}(0)$; that is, $a_3 = \frac{1}{3!}f^{(3)}(0)$.

Continuing in this way, we find that we must have $a_4 = \frac{1}{4!}f^{(4)}(0)$, $a_5 = \frac{1}{5!}f^{(5)}(0)$, and so on, until finally, to ensure that $p^{(n)}(0) = f^{(n)}(0)$, we must have $a_n = \frac{1}{n!}f^{(n)}(0)$.

The resulting formula for the polynomial $p(x)$ is given below.

Taylor polynomials about 0

Let f be a function that is n-times differentiable at 0. The **Taylor polynomial** of degree n about 0 for f is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \quad (2.1)$$

The formula allows us to find the Taylor polynomial $p(x)$ of degree n about 0 for any suitable function f, by calculating the value of f at 0, and the values of the first, second, third, ..., nth derivatives of f at 0, and substituting these into the formula.

Formula (2.1) confirms that any Taylor polynomial about 0 for a function f can be obtained from a Taylor polynomial of lower degree about 0 for f by adding the appropriate further terms. For example, the Taylor polynomial of degree 1 for f about 0 is

$$f(0) + f'(0)x,$$

while the Taylor polynomial of degree 2 for f about 0 is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2.$$

For some functions f the value of $f^{(n)}(0)$ is 0. In such a case the formula gives a polynomial whose degree is less than n. If this happens, then the polynomial is still referred to as the Taylor polynomial of degree n about 0 for f. This means that a Taylor polynomial of degree n is not necessarily a polynomial of degree n. You have seen an example of this already, in the case $n = 2$, and you will see more examples later in this section.

See Activity 1.6.

Although all the Taylor polynomials that you have seen until now have had degree 1 or above, you can take $n = 0$ in formula (2.1) to obtain a Taylor polynomial of degree 0 about 0 for a function f. This is the constant function whose graph is the horizontal line through the point $(0, f(0))$. Taylor polynomials of degree 0 are called **constant Taylor polynomials**.

For example, the Taylor polynomial of degree 0 about 0 for exp is $p(x) = e^0$; that is, $p(x) = 1$.

You have seen the terms 'linear' and 'quadratic' used to describe Taylor polynomials of degree 1 and degree 2, respectively. The terms **cubic**, **quartic** and **quintic** will be used to refer to Taylor polynomials of degree 3, degree 4 and degree 5, respectively.

Example 2.1 A quartic Taylor polynomial for exp

Find the quartic Taylor polynomial about 0 for the function $f(x) = e^x$.

Solution

The nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all positive integers n. Hence, by formula (2.1), the required polynomial is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4;$$

that is,

$$p(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

As expected, the first three terms agree with the quadratic Taylor polynomial about 0 for the exponential function, which we found in Example 1.2.

Comment

Since $f^{(n)}(0) = 1$ for all positive integers n, it follows from formula (2.1) that for any n the Taylor polynomial of degree n about 0 for the exponential function is

$$p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

The graphs of the function $f(x) = e^x$ and the quartic Taylor polynomial about 0 found in Example 2.1 are shown in Figure 2.1. Notice how much better this approximation is than that in Figure 1.6, page 13.

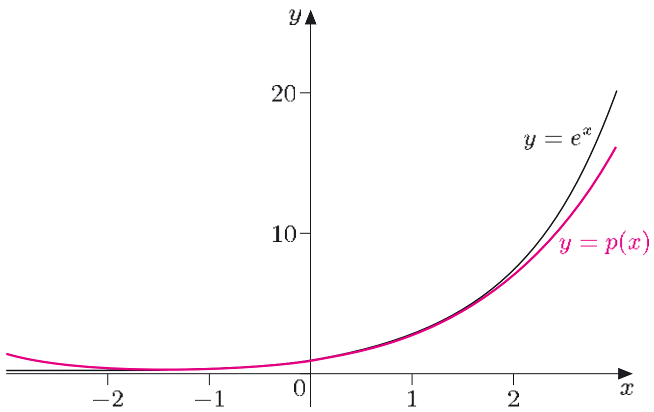


Figure 2.1 Quartic Taylor polynomial about 0 for $f(x) = e^x$

In the next activity you are asked to find two quartic Taylor polynomials.

Activity 2.1 Quartic Taylor polynomials for cos and sin

Find the quartic Taylor polynomials about 0 for each of the following functions.

(a) $f(x) = \cos x$

(b) $f(x) = \sin x$

Solutions are given on page 51.

Comment

You should check that the constant term, the term in x and the term in x^2 in your answers are the same as those in the quadratic Taylor polynomials for the functions cos and sin, which you were asked to find in Activities 1.5 and 1.6.

The quartic Taylor polynomial about 0 for the sine function is an example of a Taylor polynomial of degree n whose polynomial degree is less than n.

The graphs of the cosine and sine functions, and the quartic Taylor polynomials for these functions that you were asked to find in Activity 2.1, are shown in Figure 2.2(a) and (b), respectively.

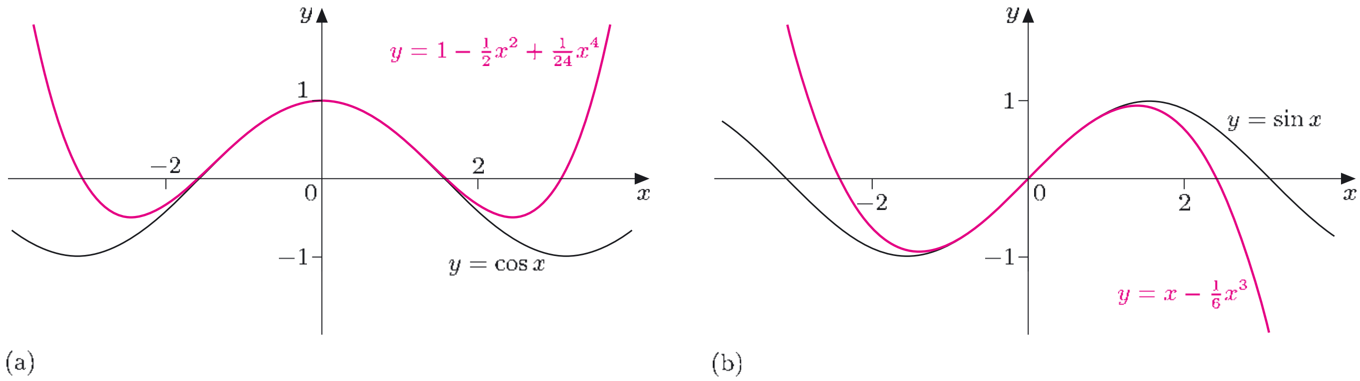


Figure 2.2 Quartic Taylor polynomials about 0 for cos and sin

In Activity 2.1 you may have noticed that the quartic Taylor polynomial about 0 for the cosine function contains terms in even powers of x only, whereas that for the sine function contains terms in odd powers of x only.

These observations are explained by the facts that cos is an even function and sin is an odd function. In fact,

Even and odd functions were introduced in Chapter C1, Subsection 3.1.

> any Taylor polynomial about 0 for an even function contains terms in even powers of x only, whereas any Taylor polynomial about 0 for an odd function contains terms in odd powers of x only.

To prove the former result, suppose that f is an even function. Thus the graph of f is unchanged under reflection in the y-axis, or, equivalently,

$$f(-x) = f(x),$$

for all x in the domain of f.

The differentiation of the left-hand side involves the Chain Rule.

If we differentiate both sides of this equation once, twice, three times, and so on, then we obtain

$$\begin{aligned} -f'(-x) &= f'(x), \\ f''(-x) &= f''(x), \\ -f^{(3)}(-x) &= f^{(3)}(x), \\ f^{(4)}(-x) &= f^{(4)}(x), \end{aligned}$$

and so on. Whenever k is odd we have

$$-f^{(k)}(-x) = f^{(k)}(x),$$

and putting $x = 0$ gives

$$-f^{(k)}(0) = f^{(k)}(0); \quad \text{that is,} \quad f^{(k)}(0) = 0.$$

Now the coefficient of x^k in the Taylor series about 0 for f is $f^{(k)}(0)/k!$, so this coefficient is zero when k is odd, and hence the Taylor series contains only terms in even powers of x, as claimed.

The result for an odd function can be proved in a similar manner, but the details are omitted here.

Activity 2.2 A Taylor polynomial of degree n

Find the Taylor polynomial of degree n about 0 for the function

$$f(x) = \frac{1}{1-x}.$$

Here you need to differentiate $f(x)$ once, twice, three times, ..., until a pattern is clear and you can write down a formula for the nth derivative.

A solution is given on page 52.

The graphs of the function $f(x) = 1/(1-x)$ and the Taylor polynomial that you were asked to find in Activity 2.2, in the case $n = 3$, are illustrated in Figure 2.3.

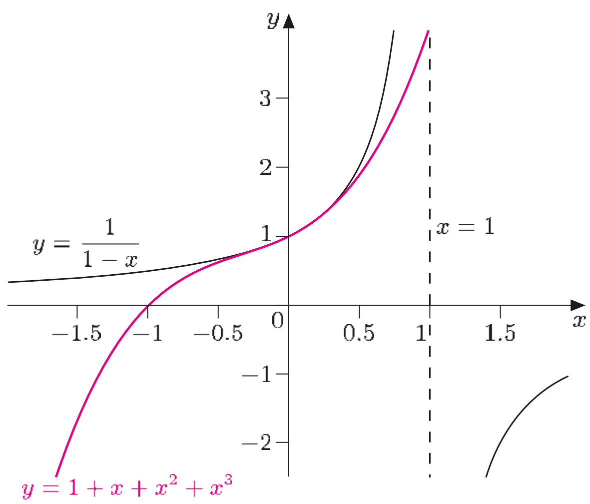


Figure 2.3 Cubic Taylor polynomial about 0 for $f(x) = 1/(1-x)$

2.2 Taylor polynomials of degree n about a

In Subsection 2.1 we considered Taylor polynomials about 0. However, it is possible to generalise the derivation of the formula for a Taylor polynomial about 0 that you saw earlier, to obtain a formula for a polynomial that approximates a suitable function f close to any chosen point a in its domain. The required formula is given below.

Here 'suitable' means that f must be differentiable at a the required number of times.

Taylor polynomials about a

Let f be a function that is n-times differentiable at a. The **Taylor polynomial** of degree n about a for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (2.2)$$

When $a = 0$, this reduces to formula (2.1) for Taylor polynomials about 0.

As in the case $a = 0$, the formula shows that any Taylor polynomial about a point a for a function f can be obtained from a Taylor polynomial of lower degree about a for f by adding the appropriate further terms. Also, again as in the case $a = 0$, it is possible for the Taylor polynomial of degree n about a for f to be a polynomial of degree less than n; this happens when $f^{(n)}(a) = 0$.

If p is a Taylor polynomial for a function f about a point a in its domain, and x is a point close to a, then $p(x)$ is an approximation for $f(x)$. Usually, the greater the degree of the Taylor polynomial, and the closer x is to a, the more accurate the approximation.

Example 2.2 A Taylor polynomial about 1 for ln

Find the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$.

Solution

The first four derivatives of the function $f(x) = \ln x$ are as follows:

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{3\times 2}{x^4} = -\frac{3!}{x^4}.$$

The product 3×2 has been simplified to 3! rather than 6 here to highlight the patterns emerging.

Evaluating f and its first four derivatives at 1 gives:

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f^{(3)}(1) = 2, \quad f^{(4)}(1) = -3!.$$

So, by formula (2.2), the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$ is

$$0 + 1 \times (x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4;$$

that is,

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

Notice that we do not multiply out the brackets in this expression. We usually leave a Taylor polynomial about a point a as a sum of terms each of which is the product of a constant and a power of $x - a$.

Comment

If you look at the patterns in $f(x), f'(x), f''(x), f^{(3)}(x), \ldots$ and $f(1), f'(1), f''(1), f^{(3)}(1), \ldots$ in the solution, then you will see that the pattern of terms continues as the degree of the Taylor polynomial increases. Thus the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n.$$

The expression $(-1)^{n-1}$ in the final term is just a neat way to give the term a negative sign when n is even and a positive sign when n is odd.

The graphs of the function $f(x) = \ln x$ and the quartic Taylor polynomial about 1 found in Example 2.2 are shown in Figure 2.4.

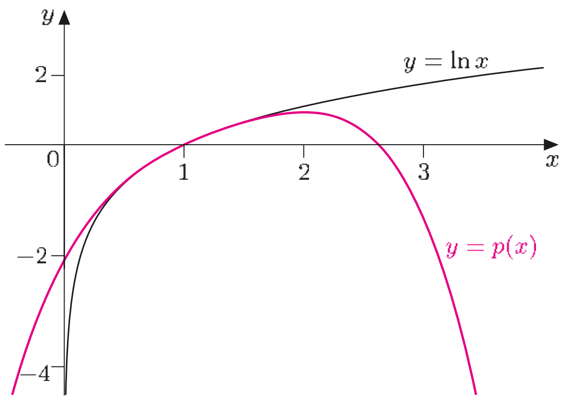


Figure 2.4 Quartic Taylor polynomial about 1 for $f(x) = \ln x$

Activity 2.3 A Taylor polynomial about $\frac{1}{6}\pi$ for sin

Find the cubic Taylor polynomial about $\frac{1}{6}\pi$ for the function $f(x) = \sin x$.

A solution is given on page 52.

Comment

The Taylor polynomial about $\frac{1}{6}\pi$ for the function $f(x) = \sin x$ contains terms in $(x - \frac{1}{6}\pi)^k$ with k even as well as with k odd. This is not surprising; the discussion about even and odd functions in Subsection 2.1 applies to Taylor polynomials about 0 only.

The graphs of the function $f(x) = \sin x$ and the cubic Taylor polynomial about $\frac{1}{6}\pi$ found in Activity 2.3 are shown in Figure 2.5.

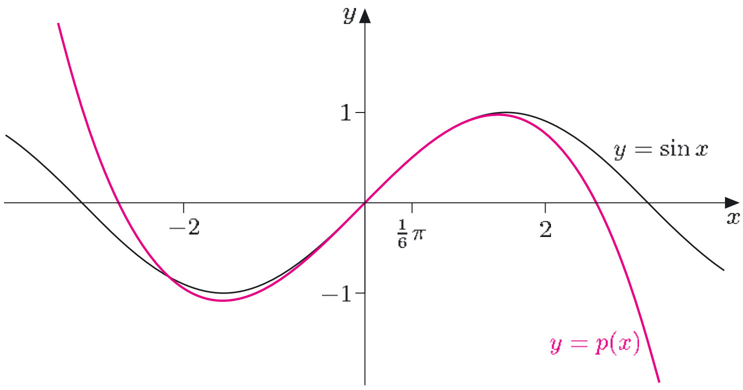


Figure 2.5 Cubic Taylor polynomial about $\frac{1}{6}\pi$ for $f(x) = \sin x$

Sigma notation

Sigma notation provides a concise way to write down Taylor polynomials. Formula (2.1) on page 17 for the Taylor polynomial of degree n about 0 for a function f is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

In sigma notation this polynomial is

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

Recall that $0! = 1$. Also note that $f^{(0)}$ is interpreted to mean f itself, and that by convention $0^0 = 1$ in series of this type.

For example, the quartic Taylor polynomial about 0 for the function $f(x) = e^x$, which was found in Example 2.1, is

$$\begin{aligned} p(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \\ &= \sum_{k=0}^{4} \frac{1}{k!}x^k. \end{aligned}$$

You are expected to use sigma notation for Taylor polynomials only in the computer-based work in Section 5.

2.3 *Using Taylor polynomials for approximation*

In this subsection Taylor polynomials are used to calculate approximations for values of functions at particular points. In doing so, it is convenient to use notation that indicates the degree of a Taylor polynomial. We do this by writing $p_n(x)$, where n is the degree, rather than just $p(x)$. Thus, for example, we can say that the Taylor polynomials of degrees 1 and 2 about 0 for the function $f(x) = e^x$ are $p_1(x) = 1 + x$ and $p_2(x) = 1 + x + \frac{1}{2}x^2$, respectively. We can also attach a subscript to the notation for the associated remainder function in a similar way; for example, $r_1(x) = f(x) - p_1(x)$ and $r_2(x) = f(x) - p_2(x)$.

It was stated in Subsection 2.2 that usually the greater the degree of a Taylor polynomial p about a point a for a function f, the more accurate $p(x)$ is as an approximation to $f(x)$ for values of x close to a. For example, Figure 2.6 shows the graphs of the Taylor polynomials of degrees 1, 2, 3 and 4 about 0 for the function $f(x) = e^x$, together with the graph of $f(x) = e^x$ itself. As expected, it appears that as the degree of the Taylor polynomial increases, its graph approximates the graph of f near 0 more and more closely.

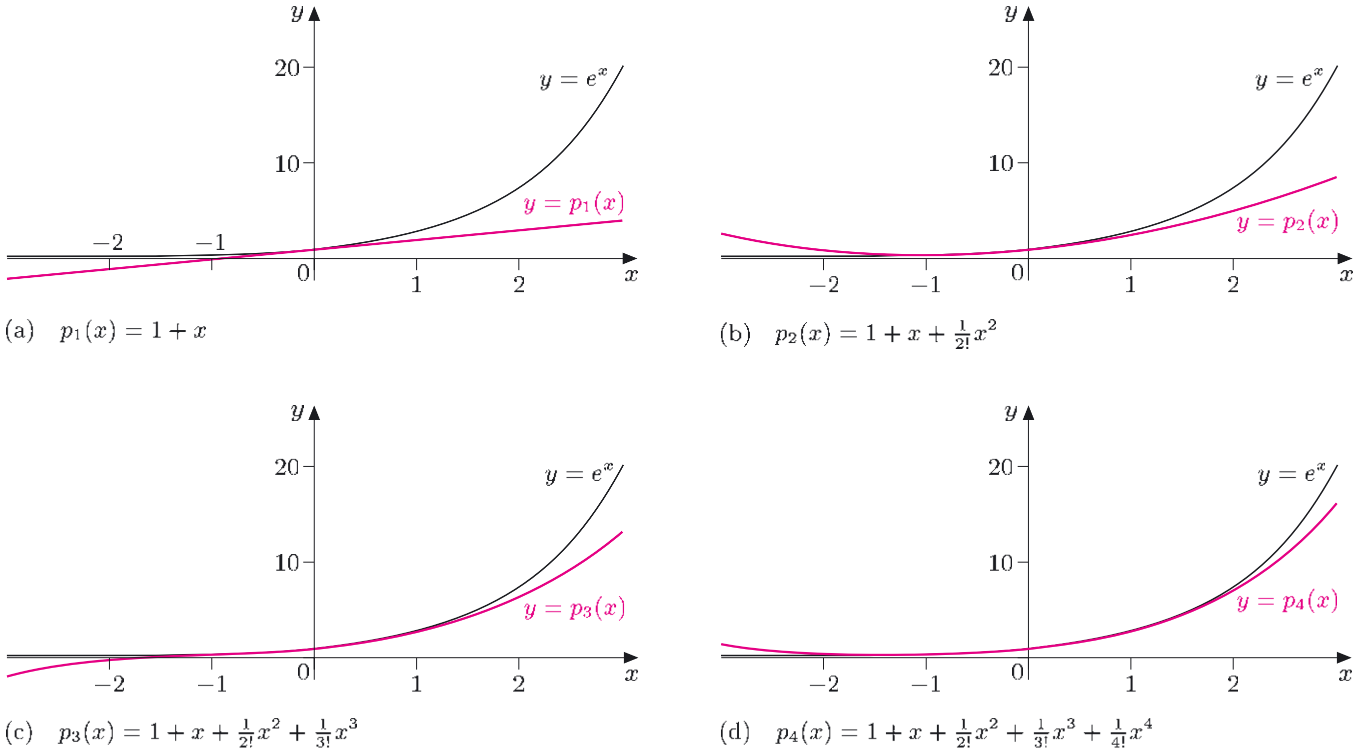


Figure 2.6 Taylor polynomials about 0 for $f(x) = e^x$

Table 2.1 provides a numerical illustration for the same function, $f(x) = e^x$, for a particular value of x near 0, namely $x = 0.25$. The value of $e^{0.25}$ is 1.284 025 4167 to ten decimal places. For values of n from 1 to 8, the table gives the Taylor polynomial $p_n(x)$ about 0 for f, the value of this polynomial when $x = 0.25$, and the associated remainder $r_n(0.25)$. All values are given to ten decimal places. You can see that as the degree n of the Taylor polynomial increases, the accuracy of $p_n(0.25)$ as an approximation for $e^{0.25}$ improves.

Table 2.1

n	$p_n(x)$	$p_n(0.25)$	$r_n(0.25) = e^{0.25} - p_n(0.25)$
1	$1 + x$	1.25	0.034 025 4167
2	$1 + x + x^2/2!$	1.281 25	0.002 775 4167
3	$1 + x + x^2/2! + x^3/3!$	1.283 854 1667	0.000 171 2500
4	$1 + x + x^2/2! + \cdots + x^4/4!$	1.284 016 9271	0.000 008 4896
5	$1 + x + x^2/2! + \cdots + x^5/5!$	1.284 025 0651	0.000 000 3516
6	$1 + x + x^2/2! + \cdots + x^6/6!$	1.284 025 4042	0.000 000 0125
7	$1 + x + x^2/2! + \cdots + x^7/7!$	1.284 025 4163	0.000 000 0004
8	$1 + x + x^2/2! + \cdots + x^8/8!$	1.284 025 4167	0.000 000 0000

As suggested by Table 2.1, Taylor polynomials can be used to calculate approximations for values of functions to any desired accuracy. If f is a function and x is a particular value in the domain of f, then to find an approximation for $f(x)$ we calculate a Taylor polynomial for f about some suitable point a close to x, and evaluate it at x. Once the Taylor polynomial has been calculated, only the standard arithmetical operations of addition, subtraction and multiplication are required to find the approximation. (Raising to a power is just repeated multiplication.)

Unfortunately there is no easy method for determining a suitable degree for the Taylor polynomial in any given case. However, there is a 'rule of thumb' that works in many cases. If we require an approximation accurate to m decimal places, then we calculate approximations using Taylor polynomials of degree 1, 2, 3, and so on, until two successive approximations agree to $m + 2$ decimal places. This method is illustrated in Example 2.3 below.

The reliability of this rule of thumb is discussed briefly in the next subsection.

Note that when we say that two numbers agree to a given number of decimal places, we mean that the values resulting from rounding them to that number of decimal places are equal. Thus, for example, 0.237 and 0.241 agree to two decimal places, since in each case rounding to two decimal places gives 0.24. However, 0.241 and 0.247 do not agree to two decimal places, since rounding to two decimal places gives 0.24 in the first case and 0.25 in the second.

Example 2.3 Finding an approximate value of a function

Use Taylor polynomials about 1 to evaluate ln(1.1) to four decimal places.

Solution

In the comment on Example 2.2 it was noted that the Taylor polynomial of degree n about 1 for $f(x) = \ln x$ is

$$p_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n.$$

Using this fact, and calculating values to six decimal places, we obtain

We calculate values to six decimal places because we wish to find a pair of successive values that agree to $4 + 2 = 6$ decimal places.

$$\begin{aligned}
p_1(1.1) &= 1.1 - 1 = 0.1,\\
p_2(1.1) &= p_1(1.1) - \tfrac{1}{2}(1.1-1)^2 = 0.095,\\
p_3(1.1) &= p_2(1.1) + \tfrac{1}{3}(1.1-1)^3 = 0.095\,333,\\
p_4(1.1) &= p_3(1.1) - \tfrac{1}{4}(1.1-1)^4 = 0.095\,308,\\
p_5(1.1) &= p_4(1.1) + \tfrac{1}{5}(1.1-1)^5 = 0.095\,310,\\
p_6(1.1) &= p_5(1.1) - \tfrac{1}{6}(1.1-1)^6 = 0.095\,310.
\end{aligned}$$

The values of $p_5(1.1)$ and $p_6(1.1)$ agree to six decimal places, so it is likely that

$$\ln(1.1) = 0.0953,$$

to four decimal places. (This is the case.)

In the above example each successive approximation $p_k(1.1)$ was calculated by evaluating just the final term of $p_k(x)$ with $x = 1.1$, and then adding this value to $p_{k-1}(1.1)$, the previous approximation. This is an efficient way to proceed, but when you work through a similar example yourself, you must make sure that each time you add an evaluated term to the previous approximation, you use a version of the previous approximation to full calculator accuracy, rather than the rounded version that you have just written down. Not doing so will cause errors in some cases.

One convenient technique for carrying out a procedure like that in Example 2.3 is to first set your calculator's memory to zero, and then work out each successive approximation by evaluating the appropriate term and adding it to your calculator's memory. After each such addition you can check the memory, round off this approximation and write it down.

The next example is similar to Example 2.3, but it involves Taylor polynomials about 0 for the sine function. Since in this case each Taylor polynomial of even degree is identical to the Taylor polynomial of degree one less (that is, $p_2(x) = p_1(x)$, $p_4(x) = p_3(x)$, and so on), we would quickly find two successive approximations that agree to any specified number of decimal places, but this would tell us nothing about the accuracy of the approximation! For this reason we consider only the Taylor polynomials of odd degree.

This is because sin is an odd function; see the discussion after Activity 2.1.

Example 2.4 Finding an approximate value of an odd function

Use Taylor polynomials about 0 to find the value of $\sin(0.2)$ to six decimal places.

Solution

From Activity 2.1(b), the quartic Taylor polynomial about 0 for the sine function is

$$p_4(x) = x - \frac{1}{3!}x^3.$$

We can use formula (2.1) on page 17 to calculate further terms to obtain Taylor polynomials of higher degree. For example, the Taylor polynomial of degree 7 about 0 for the sine function is

$$p_7(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7.$$

Using this fact, and calculating values to eight decimal places, we obtain

We calculate values to $6 + 2 = 8$ decimal places.

$$p_1(0.2) = 0.2,$$

$$p_3(0.2) = p_1(0.2) - \frac{1}{3!}(0.2)^3 = 0.198\,666\,67,$$

$$p_5(0.2) = p_3(0.2) + \frac{1}{5!}(0.2)^5 = 0.198\,669\,33,$$

$$p_7(0.2) = p_5(0.2) - \frac{1}{7!}(0.2)^7 = 0.198\,669\,33.$$

The values of $p_5(0.2)$ and $p_7(0.2)$ agree to eight decimal places, so it is likely that

$$\sin(0.2) = 0.198\,669,$$

to six decimal places. (This is the case.)

In the next activity you are asked to find approximate values of the exponential and cosine functions at particular points in their domains.

Activity 2.4 Finding approximate values of functions

Use Taylor polynomials about 0 to find

(a) the value of $e^{-0.05}$ to four decimal places;

(b) the value of $\cos(0.2)$ to six decimal places.

Solutions are given on page 52.

2.4 Accuracy of approximations from Taylor polynomials

The box below contains a useful result on the accuracy of approximations found by evaluating Taylor polynomials. Its proof is beyond the scope of the course. Unfortunately the result and its use is quite complicated, and it has been included only for your interest.

This subsection will not be assessed.

A formula for the remainder function

Let f be a function that is $(n+1)$-times differentiable on an open interval I containing the point a. Let p_n be the Taylor polynomial of degree n about a for f, and let r_n be the associated remainder function; that is, $r_n(x) = f(x) - p_n(x)$. Then, for each number x in the interval I, we have

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \tag{2.3}$$

for some number c between a and x.

If you continue your studies of mathematics, then you will probably meet this result again in the context of similar results, where its form will seem more natural. It is known as Taylor's Theorem.

Notice that formula (2.3) for $r_n(x)$ is just the formula for the term in $(x-a)^{n+1}$ in the Taylor polynomial of degree $n+1$ about a for f (that is, the 'next term'), but with a replaced by c in the expression for the coefficient of $(x-a)^{n+1}$.

It is important to appreciate that the number c in the formula depends on x (as well as on f and n). In general different values of c are needed for different values of x. Since we do not know which value of c is required for any given value of x, we cannot use the formula directly. Despite this, it is more useful than you might think: we can often use it to show that the magnitude of $r_n(x)$ is less than a certain value, and this tells us about the accuracy of the approximation provided by the Taylor polynomial $p_n(x)$.

For example, consider the Taylor polynomial of degree 4 about 0 for the function $f(x) = \sin x$, which is $x - \frac{1}{6}x^3$. The formula for the remainder function in the case of this function and its quartic Taylor polynomial is

$$r_4(x) = \frac{f^{(5)}(c)}{5!} \times (x-0)^5 = \frac{x^5 \cos c}{120},$$

for some value of c between 0 and x.

Repeated differentiation shows that $f^{(5)}(x) = \cos x$.

To obtain an approximation for $\sin(0.3)$, we take $x = 0.3$ in the Taylor polynomial, which gives

$$\sin(0.3) \simeq 0.3 - \frac{1}{6}(0.3)^3 = 0.2955.$$

Taking $x = 0.3$ in the formula for the remainder shows that the difference between the value of $\sin(0.3)$ and this approximation is

$$r_4(0.3) = \frac{(0.3)^5 \cos c}{120} = 0.000\,020\,25 \cos c,$$

for some value of c between 0 and 0.3. We do not know the exact value of c, but we do know that $|\cos c| \leq 1$, so certainly

$$|r_4(0.3)| = 0.000\,020\,25\, |\cos c| \leq 0.000\,020\,25,$$

which tells us that the approximation 0.2955 for $\sin(0.3)$ is accurate to at least four decimal places.

Formula (2.3) for the remainder function can be used to show that the 'rule of thumb' of the previous subsection is valid in most cases that you are likely to come across.

Summary of Section 2

This section has introduced:

- ◇ the Taylor polynomial of degree n for a function f about any point a in its domain;
- ◇ the fact that a Taylor polynomial of degree n about a for a function f approximates f for values of x close to a, and that usually the accuracy of the approximation decreases the further x is from a;
- ◇ the fact that a Taylor polynomial $p(x)$ about 0 for an even function contains terms in even powers of x only, and a Taylor polynomial $p(x)$ about 0 for an odd function contains terms in odd powers of x only;
- ◇ the fact that the accuracy of the approximation provided by a Taylor polynomial usually increases as its degree n increases;
- ◇ a 'rule of thumb' for using Taylor polynomials to find an approximation for a value of a function.

Exercises for Section 2

Exercise 2.1

Find the quintic Taylor polynomial for each of the following functions about the given point.

(a) $f(x) = x^6$ about 1 (b) $f(x) = \ln(1 - x)$ about 0

(c) $f(x) = \sin x$ about $\frac{1}{4}\pi$ (d) $f(x) = (1 + x)^{1/2}$ about 0

Exercise 2.2

By expressing 10 as $9(1 + \frac{1}{9})$ and using the result of Exercise 2.1(d), evaluate $\sqrt{10}$ to three decimal places.

3 Taylor series

In Section 2 you saw that usually the greater the degree of a Taylor polynomial about a point a for a function f, the more accurate the Taylor polynomial is as an approximation for f close to a. But what happens if we take a Taylor polynomial of 'infinite degree'; that is, if we add on all possible terms? This question is considered in this section.

3.1 What is a Taylor series?

In Activity 2.2 you saw that the Taylor polynomial of degree n about 0 for the function $f(x) = 1/(1-x)$ is

$$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n. \tag{3.1}$$

Let us now consider the infinite series obtained by letting n tend to infinity, which is

$$1 + x + x^2 + x^3 + \cdots . \tag{3.2}$$

If an infinite series has the property that, as we add on more and more terms, the sequence of sums obtained settles down in the long term to a particular value s, then we say that s is the **sum** of the series. From what we saw in Section 2, as n becomes large, we expect $p_n(x)$ in equation (3.1) to settle down to $1/(1-x)$, at least for values of x near 0. That is, we expect the sum of the series (3.2) to be $1/(1-x)$ for such x. But for precisely which values of x does this happen?

We also say that the series is *equal* to s.

The answer, which will be verified shortly, is that series (3.2) sums to $1/(1-x)$ for all values of x in the range $-1 < x < 1$. For example, with $x = 0.5$ the series

$$1 + 0.5 + (0.5)^2 + (0.5)^3 + \cdots = 1 + 0.5 + 0.25 + 0.125 + \cdots$$

has sum

$$\frac{1}{1-0.5} = 2.$$

For values of x outside the range $-1 < x < 1$, the series does not sum to $1/(1-x)$; in fact, it does not have a finite sum for such x. For example, if $x = 2$ then the function $f(x) = 1/(1-x)$ has value -1, but the series is

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = 1 + 2 + 4 + 8 + 16 + \cdots ,$$

which does not sum to any real number, since as more and more terms are added on, the sum increases without limit.

We can confirm that series (3.2) sums to $1/(1-x)$ for all x in the range $-1 < x < 1$ by noting that it is an infinite geometric series; that is, it is of the form $a + ar + ar^2 + ar^3 + \cdots$. By a standard result, such a series has sum $a/(1-r)$, provided that $|r| < 1$. Series (3.2) is of this form with $a = 1$ and $r = x$, so it sums to $1/(1-x)$, provided that $|x| < 1$; that is, provided that $-1 < x < 1$, as claimed above.

This result was given in MST121 Chapter B1, Subsection 5.3.

If f is a function that is differentiable infinitely many times at a point a in its domain, then the infinite series obtained by letting the degree n of a Taylor polynomial about a for f tend to infinity is called the **Taylor series** about a for f.

Taylor series about a

Let f be a function that is differentiable infinitely many times at a. The **Taylor series** about a for f is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots. \quad (3.3)$$

The point a is called the **centre** of the Taylor series.

If $a = 0$, then the Taylor series reduces to

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots. \quad (3.4)$$

Taylor series and Taylor polynomials can be expressed using variables other than x. We normally use x unless there is a reason not to do so.

If x is a point for which the Taylor series in formula (3.3) sums to $f(x)$, then we say that the Taylor series is **valid** for the point x. For example, you saw at the beginning of this subsection that the Taylor series about 0 for the function $f(x) = 1/(1-x)$ is valid for all x in the range $-1 < x < 1$, but not for values of x outside this range. Any range of values of x for which a Taylor series is valid is called a **range of validity** for the series, and the series is said to *represent* the function on any range of validity. Thus $-1 < x < 1$ is a range of validity for the Taylor series about 0 for the function $f(x) = 1/(1-x)$. Hence $0 < x < \frac{1}{2}$, for example, is also a range of validity for this series (see Figure 3.1) since each such value of x lies in the range $-1 < x < 1$; but $0 < x < 3$ is not, since it includes values outside the range $-1 < x < 1$.

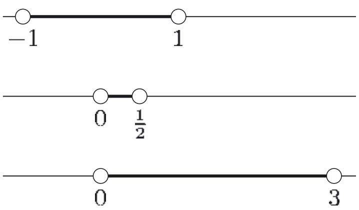


Figure 3.1 Intervals

A range of validity is a subset of the set $\mathbb{R}$ of real numbers, so it should really be denoted using appropriate set notation. For example, the range of validity $-1 < x < 1$ could be denoted by $(-1, 1)$ using the usual notation for an open interval. However, in practice the double inequality notation is convenient, and it will be used throughout this chapter.

A Taylor series about a is always valid for $x = a$. This is because if we set $x = a$ in formula (3.3), then all the terms except the first are equal to zero, so the sum of the series is just the first term $f(a)$, which is precisely the value of f at a. This is by design, since the first term of a Taylor polynomial is chosen to be $f(a)$ to ensure that the value of the polynomial at a is the same as that of f.

Example 3.1 A Taylor series for exp

Find the Taylor series about 0 for the function $f(x) = e^x$.

Solution

The nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all positive integers n. Hence, by formula (3.4), the required Taylor series is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots.$$

In the next activity you are asked to find Taylor series about 0 for the cosine and sine functions.

Activity 3.1 Finding Taylor series

Find the Taylor series about 0 for each of the following functions, writing down enough terms to make the general pattern clear.

(a) $f(x) = \cos x$

(b) $f(x) = \sin x$

In each case you should be able to see the pattern in the values $f(0), f'(0), f''(0), f^{(3)}(0), \ldots$ from your working for Activity 2.1. You are not expected to find ranges of validity for these series.

The first part of your working for Activity 2.4(b) may also be helpful.

Solutions are given on page 52.

The earlier discussion about Taylor polynomials about 0 for even and odd functions extends to Taylor *series* about 0. Thus, as you may have noticed in Activity 3.1, the Taylor series about 0 for the even function cos contains terms in even powers of x only, and that for the odd function sin contains terms in odd powers of x only.

The discussion followed Activity 2.1.

You may be surprised to learn that the Taylor series about 0 for the exponential, sine and cosine functions are all valid for every real number x. In other words, the equations

$$
\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots, \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots,
\end{aligned}
$$

are true for all $x \in \mathbb{R}$.

Thus each of these three functions is represented on its domain by its Taylor series about 0.

When you remember that the coefficients of a Taylor series for a function are chosen by taking into account the value of the function and its derivatives at a *single* point a, it may seem amazing that the resulting series can turn out to be equal to the function for *every* real number x.

All the Taylor series that you have seen so far in this section have had centre 0. In the next activity you are asked to find a Taylor series with a different centre.

Activity 3.2 A Taylor series about a point other than 0

Use formula (3.3) to find the Taylor series about 1 for the function $f(x) = \sqrt{x}$, writing down enough terms to make the general pattern clear. You are not expected to find a range of validity for this series.

A solution is given on page 53.

Comment

This Taylor series is valid for $0 < x < 2$.

The function $f(x) = \sqrt{x}$ has no Taylor series about 0, since it is not differentiable at 0. (Its derived function $f'(x) = \frac{1}{2}x^{-1/2}$ is not defined at 0.)

Sigma notation

Formula (3.3) on page 30 for the Taylor series about a for a function f can be written concisely in sigma notation as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

For example, the Taylor series about 0 for the function $f(x) = e^x$, given in Example 3.1, is

$$\sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

In the cases of odd and even functions it is more awkward to write down the Taylor series about 0 in sigma notation, because for even functions such series contain only even powers of x and for odd functions only odd powers are involved. For example, the Taylor series for the function $f(x) = \sin x$, given on page 31, is usually written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

For $k = 0, 1, 2, \ldots$, the expression x^{2k+1} equals x, x^3, x^5, $\ldots$, and the expression $(-1)^k$ looks after the alternating signs. The Taylor series for the function $f(x) = \cos x$, given on page 31, is usually written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}.$$

For $k = 0, 1, 2, \ldots$, the expression x^{2k} equals 1, x^2, $x^4, \ldots$.

Taylor series about 0 for other odd and even functions can be written in a similar way.

3.2 Some standard Taylor series

In Subsection 3.1 we obtained the Taylor series about 0 for the exponential, sine and cosine functions, and the function $f(x) = 1/(1-x)$. For ease of reference, these series are stated in the following box, along with the Taylor series about 0 for two other standard functions. These Taylor series can be obtained using formula (3.4). The box also gives ranges of validity for the Taylor series; the proof of these ranges is beyond the scope of this course.

However, we showed earlier that $-1 < x < 1$ is a range of validity for the Taylor series for $f(x) = 1/(1-x)$.

Standard Taylor series about 0

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots, \quad \text{for } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots, \quad \text{for } x \in \mathbb{R}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for } -1 < x < 1$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots, \quad \text{for } -1 < x < 1$$

Here α can be any real number.

The last series here is called the **binomial series**. Notice that each of the ranges of validity given in the box is an open interval that is symmetric about the centre 0 of the series.

These ranges of validity are the 'maximum' ranges for which the series are valid, with two exceptions. The series for $\ln(1+x)$ is also valid when $x = 1$, but the box gives the range $-1 < x < 1$ because it is often convenient to work with ranges that are open intervals. The other exception involves the binomial series. For $-1 < x < 1$, this series sums to $(1+x)^\alpha$ for any real number α, including negative and fractional numbers. For most values of α, the maximum range of validity is $-1 < x < 1$, but when α is a non-negative integer, the series is valid for *every* real number x. You will see after the next example why this is so.

The resulting series

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

was given in MST121 Chapter B1, Subsection 5.4.

The binomial series was discovered by Isaac Newton, as discussed in the video band 'The birth of calculus' associated with Chapter C1.

The box gives a Taylor series about 0 for the function $\ln(1+x)$, rather than for the standard function $\ln x$. This is because the function $\ln x$ has no Taylor series about 0, since its domain does not include 0.

The domain of ln is $(0, \infty)$.

Example 3.2 Finding binomial series

Use the binomial series to find the Taylor series about 0 for each of the following functions. In each case state a range of validity for the series.

(a) $f(x) = 1/(1+x)$ (b) $f(x) = (1+x)^4$

Solution

(a) Since $1/(1+x) = (1+x)^{-1}$, we can take $\alpha = -1$ in the binomial series to give

$$\begin{aligned}\frac{1}{1+x} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots .\end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

(b) Taking $\alpha = 4$ in the binomial series, we obtain

$$\begin{aligned}(1+x)^4 &= 1 + 4x + \frac{4\times 3}{2!}x^2 + \frac{4\times 3\times 2}{3!}x^3 + \frac{4\times 3\times 2\times 1}{4!}x^4 \\ &\quad + \frac{4\times 3\times 2\times 1\times 0}{5!}x^5 + \frac{4\times 3\times 2\times 1\times 0\times(-1)}{6!}x^6 + \cdots \\ &= 1 + 4x + 6x^2 + 4x^3 + x^4.\end{aligned}$$

This Taylor series is valid for $-1 < x < 1$.

In fact it is valid for $x \in \mathbb{R}$, which would be an equally appropriate answer. See the discussion below.

Example 3.2(b) illustrates that it is possible for a Taylor series to have a *finite* number of terms. This occurs when the coefficients of all terms from some term onwards are zero. The series in this example probably also looks rather familiar to you: it is the binomial expansion of $(1+x)^4$.

Binomial expansions, given by the Binomial Theorem, were introduced in Chapter B1, Subsection 5.1. The binomial series can be used to find a series for any expression of the form $(c+x)^\alpha$, as you will see in Section 4.

If α is any positive integer, then all terms after the term in x^α in the binomial series for $(1+x)^\alpha$ contain the factor $\alpha - \alpha = 0$ and are therefore equal to zero. The series is then the same as the binomial expansion of $(1+x)^\alpha$, which is valid for all $x \in \mathbb{R}$. The binomial series therefore generalises the binomial expansion of $(1+x)^\alpha$ from cases where α is a positive integer to cases where α can be any real number.

Activity 3.3 Finding a binomial series

Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1+x)^{1/2}$. (Write down enough terms to make the general pattern clear.) State a range of validity for the series.

A solution is given on page 53.

To *truncate* a series at a term is to delete all subsequent terms.

Once a Taylor series for a function f is known, we can find the corresponding Taylor polynomial of any degree n by truncating the series at an appropriate term.

Activity 3.4 Taylor polynomials from Taylor series

Using the Taylor series about 0 for the function $f(x) = \ln(1+x)$, write down the cubic Taylor polynomial about 0 for f.

A solution is given on page 53.

The Taylor polynomials obtained by truncating a Taylor series for a function f can, in principle, be used to find approximations for $f(x)$ for all values of x for which the series is valid. However, in general, the further x is from the centre a of the series, the greater the degree of the Taylor polynomial needed to provide a particular level of accuracy, so the method is usually not practicable for distant values of x.

For example, you have seen that the Taylor series about 0 for the function $f(x) = \ln(1+x)$ is valid for all values of x in the range $-1 < x < 1$. This means that, in principle, Taylor polynomials about 0 can be used to find an approximation for $\ln(1+x)$ for any value of x in the interval $(-1, 1)$.

Thus such polynomials can be used to find an approximation for $\ln x$ for any value of x in the interval $(0, 2)$.

The method that you saw in Subsection 2.3 for finding an approximation for a value $f(x)$ of a function f is straightforward to carry out if you know the Taylor series for f about a point a close to x, since you can obtain the required Taylor polynomials by truncating the Taylor series at the appropriate term. You are asked to do this in the next activity.

Activity 3.5 Finding an approximate value for a function

Notice that $x = 0.1$ lies within the range of validity $-1 < x < 1$ for the Taylor series about 0 for the function $f(x) = (1+x)^{1/2}$.

By writing 1.1 as $1 + 0.1$, use the series found in Activity 3.3 to find a value for $\sqrt{1.1}$ to three decimal places.

A solution is given on page 53.

All the standard Taylor series given in the box on page 32 have centre 0. Taylor series (and Taylor polynomials) about 0 are often called **Maclaurin series** (and **Maclaurin polynomials**). For example, rather than saying that $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ is the Taylor series about 0 for the function $f(x) = \ln(1+x)$, we can say that it is the Maclaurin series for the function $f(x) = \ln(1+x)$, from which it is understood that the centre is 0.

In this course, such series are referred to as Taylor series about 0, rather than Maclaurin series.

Taylor series and Taylor polynomials are named after the English mathematician Brook Taylor (1685–1731), who was educated at home and then at St John's College, Cambridge. In 1715 he published *Methodus Incrementorum Directa et Inversa*, which includes the work on which this chapter is based, as well as the technique for integration by parts which you studied in Chapter C2. The importance of Taylor polynomials remained largely unrecognised until much later in the eighteenth century. Taylor was elected as a Fellow of the Royal Society in 1712, and appointed to the committee for adjudicating the claims of Newton and Leibniz to have invented the calculus. He also wrote works on perspective and was a talented musician and artist.

Taylor was not, in fact, the first mathematician to discover Taylor series. The Scottish mathematician James Gregory (1638–1675) discovered them more than forty years before Taylor, and several other mathematicians, including Newton and Leibniz, also independently discovered versions of them before Taylor published his work.

Colin Maclaurin (1698–1746) was born in Argyllshire, studied at the University of Glasgow and became professor of mathematics first at Marischal College, Aberdeen and then at the University of Edinburgh. In 1742 he published the two volume *Treatise of Fluxions*, which was the first systematic exposition of Newton's methods in calculus. He wrote it as a reply to attacks made on the calculus for its lack of rigorous foundations. In this treatise Maclaurin uses Taylor series about 0. Although he acknowledged Taylor, Maclaurin's name is now often used to describe these series.

Summary of Section 3

This section has introduced:

- ◇ the Taylor series for a function f about any point a in its domain;
- ◇ the idea of a range of validity for a Taylor series for a function;
- ◇ the Taylor series about 0 for several standard functions.

Exercises for Section 3

Exercise 3.1

Use the definition of a Taylor series to find the Taylor series about 2 for the function $f(x) = x^{-1}$. You are not expected to find a range of validity for this series.

Exercise 3.2

Use the binomial series to find the Taylor series about 0 for the function $f(x) = (1 + x)^5$, and state a range of validity for this series.

Exercise 3.3

Use the binomial series to find the cubic Taylor polynomial about 0 for the function $f(x) = 1/\sqrt[3]{1 + x}$, simplifying each coefficient.

Exercise 3.4

By writing 1.25 as $1 + 0.25$, use the Taylor series for the function $f(x) = \ln(1 + x)$ about 0 to find a value for $\ln(1.25)$ to three decimal places.

4 Manipulating Taylor series

In this section you will see some methods that allow us to obtain Taylor series for many functions from a few known Taylor series such as the standard ones on page 32. This usually involves much less work than obtaining the required Taylor series directly, using formula (3.3) or (3.4) on page 30.

When finding a Taylor series for a function, you may make use of any of the standard Taylor series. You are not expected to derive any of the standard series unless explicitly asked to do so.

4.1 Substituting for the variable in a Taylor series

You have seen that the Taylor series about 0 for the function $g(x) = 1/(1-x)$ is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Let us consider the effect of substituting $x = 2t$ in this equation. We obtain

$$\begin{aligned}\frac{1}{1-2t} &= 1 + 2t + (2t)^2 + (2t)^3 + \cdots \\ &= 1 + 2t + 4t^2 + 8t^3 + \cdots.\end{aligned}$$

We have obtained a series equal to $1/(1-2t)$. Since the Taylor series for $g(x) = 1/(1-x)$ is valid for $-1 < x < 1$, the above series in t is equal to $1/(1-2t)$ for $-1 < 2t < 1$, that is, for $-\frac{1}{2} < t < \frac{1}{2}$.

We now replace t by x (since it is more usual to use x rather than t for the variable) to give

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots, \quad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}. \tag{4.1}$$

The series in equation (4.1) is equal to the function $f(x) = 1/(1-2x)$ for $-\frac{1}{2} < x < \frac{1}{2}$, but is it a *Taylor* series? It is of the right form to be a Taylor series about 0, since each of its terms is a constant mutiplied by a power of x. However, if we were to use formula (3.4) on page 30 to find the Taylor series about 0 for f, would we obtain the same series? The answer to this question is yes. We have the following result.

The proof of this result is beyond the scope of this course.

> Let f be a real function f. If we can find by any means a series
>
> $$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots \tag{4.2}$$
>
> that is equal to $f(x)$ for all x in some open interval containing a, then this series is the Taylor series about a for f, and hence it is the *only* series of this form that is equal to $f(x)$ for all x in that interval.

This result will be assumed throughout the rest of this section.

Taylor series can be found for many functions f by substituting for the variable in a known Taylor series. A range of validity for the new series can often be deduced from a range of validity for the original Taylor series. When substituting for the variable, it is quicker to avoid introducing a new variable t as we did above, and instead to replace x in the original Taylor series by an expression involving x, as illustrated in the next example.

Example 4.1 Substituting into a Taylor series

Find the Taylor series about 0 for the function

$$f(x) = \frac{1}{1+x^2},$$

and determine a range of validity for this series.

Solution

The Taylor series about 0 for $1/(1-x)$ is, from page 32,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

Therefore the Taylor series about 0 for $1/(1+x^2) = 1/(1-(-x^2))$ is

$$\begin{aligned} \frac{1}{1+x^2} &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \qquad (4.3) \\ &= 1 - x^2 + x^4 - x^6 + \cdots. \end{aligned}$$

Here we replace x by $-x^2$, which is equivalent to making the substitution $x = -t^2$ and then replacing t by x. Also note that in this solution and elsewhere it is often convenient to refer to a function by just giving its rule rather than by assigning a name.

The Taylor series for $1/(1-x)$ is valid for $-1 < x < 1$, so the series for $1/(1-(-x^2))$ is valid for $-1 < -x^2 < 1$. The left-hand inequality is $-1 < -x^2$, which is equivalent to $1 > x^2$; that is, $-1 < x < 1$. The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ and therefore does not place any restriction on x, since the square of any real number is non-negative. Thus the Taylor series for $1/(1+x^2)$ is valid for $-1 < x < 1$.

When you have to solve a 'double' inequality like $-1 < -x^2 < 1$, it is often helpful to split it into two single inequalities and solve each independently, as was done in Example 4.1. With simple inequalities, such as $-1 < 2x < 1$, you may be able to solve both together; for example, in this case we simply multiply both inequalities by $\frac{1}{2}$ to obtain $-\frac{1}{2} < x < \frac{1}{2}$.

When you replace x in a Taylor series by an expression involving x, it is helpful to enclose the whole expression in brackets at each replacement, and then simplify the resulting terms, as illustrated in Example 4.1. Make sure that you enclose the *whole* expression in brackets; for instance, in equation (4.3) the third term is $(-x^2)^2 = x^4$, *not* $-(x^2)^2 = -x^4$.

Activity 4.1 Substituting into a Taylor series

By substituting for the variable in a standard Taylor series, find the Taylor series about 0 for each of the following functions. In each case so far determine a range of validity for the series.

Although the series in part (a) can be found by taking $\alpha = -1$ in the binomial series, as was done in Example 3.2(a), it is more convenient to make a suitable substitution in the Taylor series for $1/(1-x)$, and this is what you are asked to do here.

(a) $f(x) = 1/(1+x)$ (b) $f(x) = \ln(1-x)$

(c) $f(x) = \ln(1+3x)$ (d) $f(x) = e^{x^3}$

Solutions are given on page 53.

You have seen that substituting for the variable in a Taylor series gives a Taylor series for another function. In each case so far, the new series has the same centre as the original series. However some substitutions can lead to a new Taylor series with a different centre. This is illustrated in the following activity.

Activity 4.2 Changing the centre of a Taylor series

You were asked to find the Taylor series about 0 for $1/(1+x)$ in Activity 4.1(a).

By replacing x by $x-1$ in the Taylor series about 0 for $1/(1+x)$, find the Taylor series about 1 for the function $f(x)=1/x$. Determine a range of validity for this series.

A solution is given on page 54.

Comment

We know that the series found in the solution is the Taylor series about 1 for $1/x$ because of the result on page 36.

4.2 Adding, subtracting and multiplying Taylor series

Another way to find Taylor series for some functions f is by applying standard arithmetic operations term by term to known Taylor series.

For example, we know that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R},$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for } -1 < x < 1.$$

It follows that

$$\begin{aligned} e^x + \frac{1}{1-x} &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ &\quad + \left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\ &= (1+1) + (1+1)x + \left(\frac{1}{2!} + 1\right)x^2 \\ &\quad + \left(\frac{1}{3!} + 1\right)x^3 + \left(\frac{1}{4!} + 1\right)x^4 + \cdots \\ &= 2 + 2x + \tfrac{3}{2}x^2 + \tfrac{7}{6}x^3 + \tfrac{25}{24}x^4 + \cdots, \end{aligned}$$

at least for $-1 < x < 1$. The series above is therefore the Taylor series about 0 for the function $f(x) = e^x + 1/(1-x)$; it is valid for $-1 < x < 1$.

Any two Taylor series with the same centre can be added or subtracted term by term in a similar manner. The resulting Taylor series is valid for all values of x for which *both* original Taylor series are valid. It may also be valid for further values of x.

Activity 4.3 Subtracting Taylor series

Find the Taylor series about 0 for the function

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x),$$

and determine a range of validity for this series.

You were asked to find the Taylor series about 0 for $\ln(1-x)$ in Activity 4.1(b).

A solution is given on page 54.

Comment

It can be shown that every positive real number t can be expressed in the form $t = (1+x)/(1-x)$ for some x in the range $-1 < x < 1$. Thus the Taylor series in this activity can be used to find an approximation for $\ln t$ for any t in the domain $(0, \infty)$ of ln. In contrast, the series for $\ln(1+x)$ can be used to find approximations for $\ln t$ only for $0 < t \leq 2$, since these are the only values of t that can be expressed in the form $t = 1 + x$ for some x in the range $-1 < x \leq 1$. For both series, the further x is from 0, the more terms of the series have to be evaluated in order to obtain the desired accuracy.

This equation can be rearranged as $x = 1 - 2/(t+1)$, which gives a suitable x for any given positive t. For example, if $t = 3$, then $x = \frac{1}{2}$.

You have seen that Taylor series can be added and subtracted. A Taylor series can also be multiplied term by term by a non-zero constant. The resulting series is valid for every value of x for which the original Taylor series is valid.

For example, we can multiply the Taylor series for e^x by 3 to deduce that

$$3e^x = 3 + 3x + \frac{3}{2!}x^2 + \frac{3}{3!}x^3 + \frac{3}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Recall that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots.$$

In the next activity you are asked to use the technique of multiplying a Taylor series by a constant, together with substitution, to find the Taylor series about 0 for the function $1/(3+x)^2$.

Activity 4.4 A Taylor series for $1/(3+x)^2$

(a) Use the binomial series to find the Taylor series about 0 for $1/(1+x)^2$, and a range of validity for this series.

(b) By using the fact that

$$\frac{1}{(3+x)^2} = \frac{1}{3^2} \times \frac{1}{(1+x/3)^2},$$

find the Taylor series about 0 for $1/(3+x)^2$, and determine a range of validity for this series.

Solutions are given on page 54.

Comment

The Taylor series about 0 for any function of the form $(c+x)^\alpha$ can be deduced from the series for $(1+x)^\alpha$ in a similar way; that is, by first expressing $(c+x)^\alpha$ as $c^\alpha(1+x/c)^\alpha$.

There are various pronunciations in use for the functions sinh and cosh; the most common ones are 'shine' or 'sine-sh' for sinh and simply 'cosh' for cosh.

You may have come across the functions sinh and cosh in your study of mathematics (they are mentioned in MST121 Chapter C2, for example), or you may have noticed their presence on your calculator. These are called **hyperbolic functions**; they can be defined in terms of the exponential function by the equations

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

and

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

We can use these equations, together with the Taylor series for e^x, to find the Taylor series for $\sinh x$ and $\cosh x$, as you will see shortly. The graphs of sinh and cosh are shown in Figure 4.1.

There are other hyperbolic functions, such as tanh, which is defined by $\tanh x = (\sinh x)/(\cosh x)$, and usually pronounced 'tansh'.

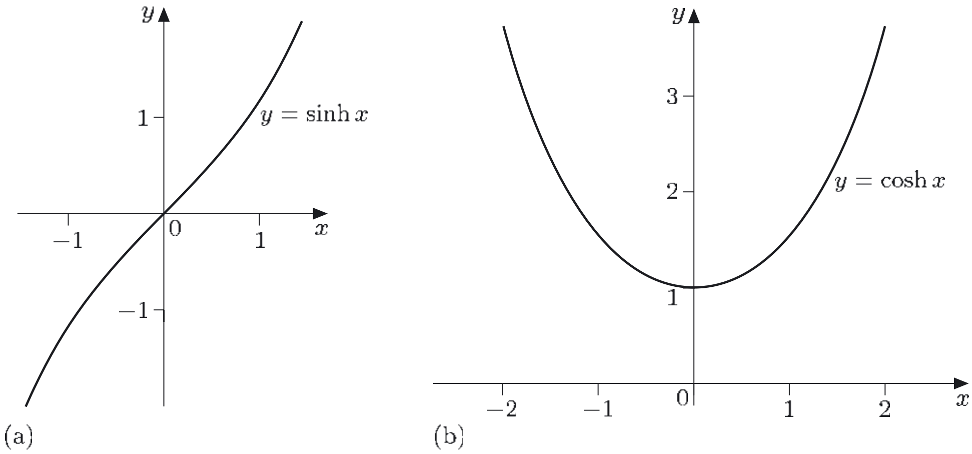


Figure 4.1 Functions sinh and cosh

Although you may not expect it from either the definitions or the graphs of sinh and cosh, these functions have many properties analogous to those of the trigonometric functions sin and cos. For example, cosh, like cos, is an even function, and sinh, like sin, is an odd function. Also the derived function of sinh is cosh and

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y, \quad \text{for all } x, y \in \mathbb{R}.$$

These properties of sinh and cosh can be verified by using the given expressions for sinh and cosh in terms of exp.

These two properties are directly analogous to properties of sin and cos. Some of the properties of sinh and cosh are similar, but not directly analogous, to those of sin and cos. For example, you have met the identity $\cos^2 x + \sin^2 x = 1$; the corresponding identity for the hyperbolic functions is $\cosh^2 x - \sinh^2 x = 1$.

The identity

$$\cosh^2 x - \sinh^2 x = 1$$

allows cosh and sinh to be used to define parametric equations for the hyperbola in standard form. This is why such functions are called hyperbolic.

The next example involves using the techniques of adding and multiplying by a constant, together with the technique of substitution, to find the Taylor series about 0 for the function cosh.

Example 4.2 A Taylor series for cosh

Find the Taylor series about 0 for the function $f(x) = \cosh x$, and determine a range of validity for this series.

Solution

We use the formula $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Now

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

On replacing x by $-x$, we obtain

$$e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots, \quad \text{for } x \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\Bigg(&\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)\\ &+ \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)\Bigg),\end{aligned}$$

for $x \in \mathbb{R}$; that is,

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

In the next activity you are asked to use a similar method to find the Taylor series about 0 for the function sinh.

Activity 4.5 A Taylor series for sinh

Find the Taylor series about 0 for the function $f(x) = \sinh x$, and determine a range of validity for this series.

A solution is given on page 54.

Notice that the Taylor series about 0 for sinh and cosh are similar to those for the corresponding trigonometric functions. The only difference is that all the coefficients in the series for $\cosh x$ and $\sinh x$ are positive, whereas the coefficients of the series for $\cos x$ and $\sin x$ alternate in sign.

You have seen that Taylor series can be added, subtracted and multiplied by a non-zero constant. We can also multiply two Taylor series together. The resulting Taylor series is valid for all values of x for which both original Taylor series are valid. It may also be valid for further values of x.

The next example illustrates the multiplication of two Taylor series. It involves finding the Taylor series about 0 of the product of a polynomial and another function. Note that it is easy to write down the Taylor series about 0 of a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

because this is already in the form of the series in formula (4.2) on page 36 (with $a = 0$, and the coefficients of all terms from that in x^{n+1} onwards equal to 0). It follows that the Taylor series about 0 of such a polynomial is the polynomial itself! It is valid for all $x \in \mathbb{R}$.

Example 4.3 Multiplying Taylor series

Find the Taylor series about 0 for the function $f(x) = (1-x)/(1+x)$, and determine a range of validity for this series.

Solution

The Taylor series for $1-x$ is $1-x$, and the Taylor series for $1/(1+x)$ is

$$1 - x + x^2 - x^3 + \cdots .$$

See Activity 4.1(a).

Therefore

$$\begin{aligned}\frac{1-x}{1+x} &= (1-x)(1-x+x^2-x^3+\cdots)\\ &= 1(1-x+x^2-x^3+\cdots) - x(1-x+x^2-x^3+\cdots)\\ &= (1-x+x^2-x^3+\cdots) - (x-x^2+x^3-\cdots)\\ &= 1-2x+2x^2-2x^3+\cdots .\end{aligned}$$

The Taylor series about 0 for $1-x$ and $1/(1+x)$ are valid for $x \in \mathbb{R}$ and for $-1 < x < 1$, respectively. Hence the above Taylor series is valid for $-1 < x < 1$.

In the next activity you are asked to carry out two multiplications of Taylor series. In each case one of the two series is a polynomial.

Activity 4.6 Multiplying Taylor series

Find the Taylor series about 0 for each of the following functions. In each case determine a range of validity for the series.

(a) $f(x) = x^2 \sin x$ (b) $f(x) = (1+x)\cos x$

Solutions are given on page 55.

Earlier you saw that a Taylor series can be multiplied by a non-zero constant. This is just a special case of the multiplication of two Taylor series. For example, the Taylor series about 0 for $3e^x$ can be obtained by multiplying together the Taylor series about 0 for the constant function $f(x) = 3$ and the Taylor series about 0 for e^x; the Taylor series about 0 for the function $f(x) = 3$ is simply 3, since 3 is a polynomial of degree 0.

In each case where we have multiplied together two Taylor series, one of the series had only finitely many non-zero terms (that is, it was a polynomial). Multiplying together two Taylor series both of which have infinitely many non-zero terms is usually a difficult task, and you will not be asked to carry out any complete multiplications of this type in this course. However, it is possible to multiply together the first few terms of two infinite Taylor series to find the first few terms of the product Taylor series. This is illustrated in the next example.

Example 4.4 Finding a Taylor polynomial by multiplication

Find the cubic Taylor polynomial about 0 for the function $f(x) = e^x \cos x$.

Solution

Using the Taylor series about 0 for e^x and for $\cos x$ (from page 32), we have

$$\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \cdots\right) \\
&= 1\left(1 - \frac{x^2}{2!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \cdots\right) \\
&\quad + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \cdots\right) + \cdots \\
&= (1 - \tfrac{1}{2}x^2 + \cdots) + (x - \tfrac{1}{2}x^3 + \cdots) + (\tfrac{1}{2}x^2 - \cdots) + (\tfrac{1}{6}x^3 - \cdots) + \cdots \\
&= 1 + x - \tfrac{1}{3}x^3 + \cdots.
\end{aligned}$$

Therefore the cubic Taylor polynomial about 0 for $f(x) = e^x \cos x$ is

$$1 + x - \tfrac{1}{3}x^3.$$

Comment

In this solution at each stage all the terms that could eventually result in terms with power 3 or less were retained, and any terms which would affect only terms with power 4 or more were ignored.

A similar method is required in the next activity.

Activity 4.7 Finding a Taylor polynomial by multiplication

Use the Taylor series about 0 for $1/(1 + x)$ and for $\sin x$ to find the cubic Taylor polynomial about 0 for the function

$$f(x) = \frac{\sin x}{1 + x}.$$

The Taylor series about 0 for $1/(1 + x)$ was found in Activity 4.1(a).

A solution is given on page 55.

4.3 Differentiating and integrating Taylor series

You have seen that the Taylor series about 0 for the function $f(x) = \sin x$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

Let us consider the effect of differentiating this series term by term, in the way that we would if it had only finitely many terms and was therefore a polynomial. We obtain the series

$$1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \cdots = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

You may recognise this as the Taylor series about 0 for $\cos x$. So by differentiating term by term the Taylor series about 0 for $f(x) = \sin x$, we have obtained the Taylor series about 0 for its derived function $f'(x) = \cos x$.

This observation suggests that term-by-term differentiation of a Taylor series about 0 for a function f yields the Taylor series about 0 for its derivative f'. This conjecture can be verified as follows.

Let f be a function that is differentiable infinitely many times at 0, and let $g = f'$. The Taylor series about 0 for f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}\,x^2 + \frac{f^{(3)}(0)}{3!}\,x^3 + \cdots + \frac{f^{(k)}(0)}{k!}\,x^k + \cdots.$$

Remember that $f(0)$, $f'(0)$, $f''(0)$, ... are constants.

Differentiating this series term by term gives the series

$$\begin{aligned} 0 &+ f'(0) + \frac{f''(0)}{2!}\,2x + \frac{f^{(3)}(0)}{3!}\,3x^2 + \cdots + \frac{f^{(k)}(0)}{k!}\,kx^{k-1} + \cdots \\ &= f'(0) + f''(0)x + \frac{f^{(3)}(0)}{2!}\,x^2 + \cdots + \frac{f^{(k)}(0)}{(k-1)!}\,x^{k-1} + \cdots. \end{aligned} \tag{4.4}$$

Since $g = f'$, we have $g(0) = f'(0)$, $g'(0) = f''(0)$, $g''(0) = f^{(3)}(0)$, and so on. Therefore the series in equation (4.4) can be written as

$$g(0) + g'(0)x + \frac{g''(0)}{2!}\,x^2 + \cdots + \frac{g^{(k-1)}(0)}{(k-1)!}\,x^{k-1} + \cdots,$$

which is the Taylor series about 0 for $g = f'$. (The general term is expressed in terms of $k-1$ instead of k.)

Taylor series can also be integrated term by term. If the Taylor series about 0 for a function f is integrated term by term, then the result is the Taylor series about 0 of an antiderivative of f. These properties of Taylor series are summarised in the following box.

Here a_0 is written for $f(0)$, a_1 for $f'(0)$, a_2 for $f''(0)/2!$, and so on, to simplify the notation.

Differentiating and integrating Taylor series about 0

Let f be a function that is differentiable infinitely many times at 0. If the Taylor series about 0 for $f(x)$ is

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$$

then the Taylor series for $f'(x)$ is

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots,$$

and the Taylor series for any antiderivative of $f(x)$ is

$$c + a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1} + \cdots,$$

where c is an arbitrary constant.

Any range of validity for the Taylor series for f which is an *open* interval is also a range of validity for the Taylor series for f' and for any antiderivative of f.

The Taylor series for an antiderivative of f may also be valid at one or both of the endpoints of the open interval, even if the Taylor series for f is not valid there.

The results in the above box can be extended to Taylor series with centres other than 0, but we shall not need to use this fact in this course.

Example 4.5 Differentiating and integrating a Taylor series

See Activity 4.1(a).

You have seen that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \quad \text{for } -1 < x < 1.$$

Use this fact to find the Taylor series about 0 for each of the following functions f. In each case state a range of validity for the series.

(a) $f(x) = 1/(1+x)^2$

(b) $f(x) = \ln(1+x)$

Solution

(a) Differentiating both sides of the given equation yields

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \cdots, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives the required Taylor series:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \quad \text{for } -1 < x < 1.$$

(b) Integrating both sides of the given equation yields

$$\ln(1+x) = c + x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1,$$

where c is an arbitrary constant. Taking $x = 0$ gives $\ln 1 = c$; hence $c = 0$. Therefore the required Taylor series is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots, \quad \text{for } -1 < x < 1.$$

This is the standard Taylor series for $\ln(1+x)$, as is to be expected. Part (b) shows the connection between this standard series and the series for $1/(1+x)$.

Comment

An alternative way to find the Taylor series in part (a) is to take $\alpha = -2$ in the binomial series, as was done in Activity 4.4(a).

The remaining activities in this subsection require you to differentiate or integrate Taylor series.

Activity 4.8 Differentiating a Taylor series

Verify that term-by-term differentiation of the Taylor series about 0 for the function $f(x) = e^x$ leaves the series unchanged.

This result corresponds to the fact that the derivative of e^x is e^x.

A solution is given on page 55.

Activity 4.9 A Taylor series for arctan

In Example 4.1 you saw that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots, \quad \text{for } -1 < x < 1.$$

Use integration to deduce the Taylor series about 0 for $\arctan x$, and state a range of validity for this series.

A solution is given on page 55.

In 1706, John Machin, who was at the University of Cambridge at the same time as Taylor, used the Taylor series about 0 for arctan to calculate the first 100 digits of π. A range of validity for this Taylor series is $-1 < x < 1$, but the series is also valid for $x = 1$. So, since $\arctan 1 = \pi/4$, it gives a representation for π as the sum of a series. Unfortunately it is not practicable to use this series to calculate π accurately because 1 is too far from the centre 0 of the series. However, Machin discovered the formula

$$\pi = 16\arctan(1/5) - 4\arctan(1/239).$$

The values $1/5$ and $1/239$ are much closer to 0 than 1 is, and hence relatively few terms need to be evaluated in order to calculate $\arctan(1/5)$ and $\arctan(1/239)$, and hence π, to 100 digits.

We obtain

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots,$$

which was given in MST121 Chapter B1, Subsection 5.4.

In the final activity of this section you are asked to use both substitution and integration to find the first few terms of the Taylor series about 0 for the inverse sine function.

Activity 4.10 A Taylor series for arcsin

(a) Use the binomial series (from page 32) to find the Taylor series about 0 for the function $1/\sqrt{1+x}$, and evaluate the coefficients of the first three terms. State a range of validity for this series.

(b) Hence, by using substitution, find the first three terms of the Taylor series about 0 for the function $1/\sqrt{1-x^2}$. Determine a range of validity for this series.

(c) By integrating the series in part (b) term by term, find the first three non-zero terms in the Taylor series about 0 for the function $f(x) = \arcsin x$, and state a range of validity for this series.

Solutions are given on page 55.

Summary of Section 4

This section has introduced:

◇ several techniques for using known Taylor series to derive further Taylor series, including substituting for the variable in a series, adding and subtracting series, multiplying a series by a constant, multiplying series together, differentiating and integrating series.

Exercises for Section 4

Exercise 4.1

Find the Taylor series about 0 for the function $f(x) = 1/(1-4x^2)$. Determine a range of validity for this Taylor series.

Exercise 4.2

By replacing x by $x-2$ in the Taylor series about 0 for $\ln(1+x)$, find the Taylor series about 2 for the function $\ln(x-1)$. Determine a range of validity for this Taylor series.

Exercise 4.3

Find the quartic Taylor polynomial about 0 for the function

$$f(x) = e^x \sin(-x),$$

evaluating each coefficient.

Exercise 4.4

(a) Find the value of x such that $1.25 = (1+x)/(1-x)$. Hence use the Taylor series found in Activity 4.3 to find a value for $\ln(1.25)$ to three decimal places.

(b) Comment on whether this method of evaluating $\ln(1.25)$ is better than the method used in Exercise 3.4.

5 Taylor series with the computer

To study this section you will need access to your computer, together with the Mathcad files for this chapter and Computer Book C.

In this section you will use the computer to plot graphs of functions and their Taylor polynomials on the same axes. This enables you to estimate a range of values over which a Taylor polynomial for a function closely approximates the function and hence estimate a range of validity for the corresponding Taylor series.

You will also learn how the computer can be used to find the first few terms of the Taylor series about 0 for a function.

Refer to Computer Book C for the work in this section.

Summary of Section 5

In this section you saw how to use computer plots to obtain ranges of values over which a function is approximated well by its Taylor polynomials, and also how the first few terms of a Taylor series about 0 are obtained using the computer.

Summary of Chapter C3

The major topics covered by this chapter are:

- ◇ the approximation of a function near a point in its domain by a Taylor polynomial;
- ◇ the representation of a function by its Taylor series;
- ◇ the manipulation of Taylor series.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

constant, linear, quadratic, cubic, quartic and quintic Taylor polynomials, Taylor polynomial of degree n, Taylor series, centre (of a Taylor series), range of validity, binomial series.

Mathematical skills

- ◇ Find Taylor polynomials and Taylor series about given points for given functions.
- ◇ Use Taylor polynomials to find approximations for values of functions.
- ◇ Use known Taylor series to find further Taylor series by substitution, addition, subtraction, multiplication, differentiation and integration.

Mathcad skills

- ◇ Find Taylor polynomials.
- ◇ Estimate a range of validity of a Taylor series, using graphs.

Ideas to be aware of

- ◇ The accuracy of the Taylor polynomial about a point a for a function f as an approximation to f usually improves both as the degree of the Taylor polynomial increases and as x approaches a.
- ◇ The polynomial degree of a Taylor polynomial of degree n may be less than n.
- ◇ An approximation for a value of a function, accurate to m decimal places, can often be found by evaluating Taylor polynomials of degrees 1, 2, 3, and so on, until two successive approximations agree to $m + 2$ decimal places.
- ◇ A Taylor series for a function f may not be equal to $f(x)$ for all x in the domain of f.
- ◇ The Taylor series (in x) about 0 for an even function contains terms in even powers of x only, and that for an odd function contains terms in odd powers of x only.

Summary of Block C

This block has introduced three key topics of calculus.

- ◇ Differentiation is the process used to find the gradient of the graph of a function f at a point on the graph, denoted by $f'(x)$ or dy/dx, and called the derivative of f. Various rules for finding derivatives were given and a graph-sketching strategy based on differentiation was introduced.
- ◇ Integration is the reverse of differentiation and it enables us to find areas under graphs. Various techniques of integration were given, and integration was also used to find volumes of solids of revolution.
- ◇ Taylor polynomials provide approximations to a given function f near a given point a in the domain of f. The values at a of the Taylor polynomial and of its derivatives, up to the nth, agree with those of the function f. The Taylor series for f about a is obtained by considering all the terms in the Taylor polynomials about a for f, and in many cases this series is equal to f on an open interval centred at a.

You will learn much more about these topics if you study further courses in mathematics.

Solutions to Activities

Solution 1.1

(a) Here $f(x) = \sin x$, so $(0, f(0)) = (0, \sin 0) = (0, 0)$. Also $f'(x) = \cos x$, so the gradient of the curve at $(0, 0)$ is $f'(0) = \cos 0 = 1$. Thus the required line passes through the point $(0, 0)$ and has gradient 1.

This line has equation $y = x$, so the linear Taylor polynomial about 0 is $p(x) = x$.

(b)

x	$f(x) = \sin x$	$p(x) = x$	$r(x) = f(x) - p(x)$
-1	-0.8415	-1	0.1585
-0.75	-0.6816	-0.75	0.0684
-0.5	-0.4794	-0.5	0.0206
-0.25	-0.2474	-0.25	0.0026
0	0	0	0
0.25	0.2474	0.25	-0.0026
0.5	0.4794	0.5	-0.0206
0.75	0.6816	0.75	-0.0684
1	0.8415	1	-0.1585

The remainder $r(x) = f(x) - p(x)$ appears to become smaller in magnitude as x approaches zero. The remainder at $x = 0$ is zero, as expected. The remainders are positive for $x < 0$ and negative for $x > 0$. (You may also have noticed that in this case the remainder for each value x is the negative of the remainder for $-x$.)

Solution 1.2

(a) Here $f(x) = \cos x$, so $(0, f(0)) = (0, \cos 0) = (0, 1)$. Also $f'(x) = -\sin x$, so the gradient of the curve at $(0, 1)$ is $f'(0) = -\sin 0 = 0$. Thus the required line passes through the point $(0, 1)$ and has gradient 0.

This line has equation $y = 1$, so the linear Taylor polynomial is $p(x) = 1$.

(b) Part (a) gives $\cos(0.2) \simeq p(0.2) = 1$.

To six decimal places, the remainder is

$$\begin{aligned} r(0.2) &= \cos(0.2) - 1 \\ &= -0.019\,933. \end{aligned}$$

Solution 1.3

(a) Here $f(x) = \ln x - 1/x$, so $(1, f(1)) = (1, \ln 1 - 1/1) = (1, -1)$. Also $f'(x) = 1/x + 1/x^2$, so the gradient of the curve at $(1, -1)$ is $f'(1) = 1/1 + 1/1^2 = 2$. Thus the required line passes through the point $(1, -1)$ and has gradient 2.

Since its gradient is 2, the line has an equation of the form $y = a_0 + 2x$. Since it passes through $(1, -1)$, we have $-1 = a_0 + 2 \times 1$; that is, $a_0 = -3$. Thus the line has equation $y = -3 + 2x$, and so the linear Taylor polynomial is $p(x) = -3 + 2x$.

(b) Here $f(x) = e^x$, so $(1, f(1)) = (1, e^1) = (1, e)$. Also $f'(x) = e^x$, so the gradient of the curve at $(1, e)$ is $f'(1) = e^1 = e$. Thus the required line passes through the point $(1, e)$ and has gradient e.

Since its gradient is e, the line has an equation of the form $y = a_0 + ex$. Since it passes through $(1, e)$, we have $e = a_0 + e \times 1$; that is, $a_0 = 0$. Thus the line has equation $y = ex$, and so the linear Taylor polynomial is $p(x) = ex$.

Solution 1.4

(a) Here $f(x) = (1 + x)^k$, so $(0, f(0)) = (0, (1 + 0)^k) = (0, 1)$. Also $f'(x) = k(1 + x)^{k-1}$, so the gradient of the curve at $(0, 1)$ is $f'(0) = k(1 + 0)^{k-1} = k$. Thus the required line passes through the point $(0, 1)$ and has gradient k.

Since its gradient is k, the line has an equation of the form $y = a_0 + kx$. Since it passes through $(0, 1)$, we have $1 = a_0 + k \times 0$; that is, $a_0 = 1$. Thus the line has equation $y = 1 + kx$, and so the linear Taylor polynomial is $p(x) = 1 + kx$.

(b) Taking $k = \frac{1}{2}$, we have

$$\sqrt{1.01} = (1 + 0.01)^{1/2} = f(0.01)$$

and

$$p(0.01) = 1 + \tfrac{1}{2} \times 0.01 = 1.005.$$

Hence $\sqrt{1.01} \simeq 1.005$. To six decimal places, the remainder is

$$\begin{aligned} r(0.01) &= f(0.01) - p(0.01) \\ &= 1.004\,988 - 1.005 \\ &= -0.000\,012. \end{aligned}$$

Solution 1.5

(a) Let the polynomial that we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$f(x) = \cos x, \qquad p(x) = a_0 + a_1 x + a_2 x^2,$$

$$f(0) = \cos 0 = 1, \quad p(0) = a_0.$$

Thus we must have $a_0 = 1$.

Next we ensure that $p'(0) = f'(0)$. We have

$$f'(x) = -\sin x, \qquad p'(x) = a_1 + 2a_2 x,$$

$$f'(0) = -\sin 0 = 0, \quad p'(0) = a_1.$$

Thus we must have $a_1 = 0$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$f''(x) = -\cos x, \qquad p''(x) = 2a_2,$$

$$f''(0) = -\cos 0 = -1, \quad p''(0) = 2a_2.$$

Thus we must have $2a_2 = -1$; that is, $a_2 = -\frac{1}{2}$. Hence the quadratic Taylor polynomial about 0 for the cosine function is

$$p(x) = 1 - \tfrac{1}{2}x^2.$$

(b) We have

$$\cos(0.2) \simeq p(0.2) = 1 - \tfrac{1}{2}(0.2)^2 = 0.98.$$

To six decimal places, the remainder is

$$\begin{aligned} r(0.2) &= \cos(0.2) - p(0.2) \\ &= 0.980\,067 - 0.98 \\ &= 0.000\,067. \end{aligned}$$

This remainder has smaller magnitude than the remainder found in Activity 1.2, so the approximation $\cos(0.2) \simeq 0.98$ is better than $\cos(0.2) \simeq 1$, as expected.

Solution 1.6

Let the polynomial that we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$f(x) = \sin x, \qquad p(x) = a_0 + a_1 x + a_2 x^2,$$

$$f(0) = \sin 0 = 0, \quad p(0) = a_0.$$

Thus we must have $a_0 = 0$.

Next we ensure that $p'(0) = f'(0)$. We have

$$f'(x) = \cos x, \qquad p'(x) = a_1 + 2a_2 x,$$

$$f'(0) = \cos 0 = 1, \quad p'(0) = a_1.$$

Thus we must have $a_1 = 1$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$f''(x) = -\sin x, \qquad p''(x) = 2a_2,$$

$$f''(0) = -\sin 0 = 0, \quad p''(0) = 2a_2.$$

Thus we must have $2a_2 = 0$; that is, $a_2 = 0$.

Hence the quadratic Taylor polynomial about 0 for the sine function is

$$p(x) = x.$$

Solution 2.1

(a) To find the quartic Taylor polynomial about 0 for $f(x) = \cos x$ we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$. We have:

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1; \\ f'(x) &= -\sin x, & f'(0) &= 0; \\ f''(x) &= -\cos x, & f''(0) &= -1; \\ f^{(3)}(x) &= \sin x, & f^{(3)}(0) &= 0; \\ f^{(4)}(x) &= \cos x, & f^{(4)}(0) &= 1. \end{aligned}$$

Hence the quartic Taylor polynomial about 0 for the cosine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\qquad + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \\ &= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4. \end{aligned}$$

(b) Similarly, to find the quartic Taylor polynomial about 0 for $f(x) = \sin x$ we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$. We have:

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0; \\ f'(x) &= \cos x, & f'(0) &= 1; \\ f''(x) &= -\sin x, & f''(0) &= 0; \\ f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1; \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0. \end{aligned}$$

Hence the quartic Taylor polynomial about 0 for the sine function is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\qquad + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{3!}x^3 \\ &= x - \tfrac{1}{6}x^3. \end{aligned}$$

Solution 2.2

To find the Taylor polynomial of degree n about 0 for $f(x) = 1/(1-x)$ we evaluate $f(0)$, $f'(0)$, $f''(0)$, $\ldots$, $f^{(n)}(0)$. The pattern is as follows:

$$\begin{aligned}
f(x) &= \frac{1}{1-x}, & f(0) &= 1;\\
f'(x) &= \frac{1}{(1-x)^2}, & f'(0) &= 1;\\
f''(x) &= \frac{2}{(1-x)^3}, & f''(0) &= 2;\\
f^{(3)}(x) &= \frac{3\times 2}{(1-x)^4} = \frac{3!}{(1-x)^4}, & f^{(3)}(0) &= 3!;\\
f^{(4)}(x) &= \frac{4\times 3\times 2}{(1-x)^5} = \frac{4!}{(1-x)^5}, & f^{(4)}(0) &= 4!;
\end{aligned}$$

and so on, terminating with

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \qquad f^{(n)}(0) = n!.$$

Hence the Taylor polynomial of degree n about 0 for $f(x) = 1/(1-x)$ is

$$\begin{aligned}
p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3\\
&\quad + \cdots + \frac{f^{(n)}(0)}{n!}x^n\\
&= 1 + x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n\\
&= 1 + x + x^2 + x^3 + \cdots + x^n.
\end{aligned}$$

Solution 2.3

To find the cubic Taylor polynomial about $\frac{1}{6}\pi$ for $f(x) = \sin x$, we evaluate $f\left(\frac{1}{6}\pi\right)$, $f'\left(\frac{1}{6}\pi\right)$, $f''\left(\frac{1}{6}\pi\right)$ and $f^{(3)}\left(\frac{1}{6}\pi\right)$, as follows:

$$\begin{aligned}
f(x) &= \sin x, & f(\tfrac{1}{6}\pi) &= \tfrac{1}{2};\\
f'(x) &= \cos x, & f'(\tfrac{1}{6}\pi) &= \tfrac{1}{2}\sqrt{3};\\
f''(x) &= -\sin x, & f''(\tfrac{1}{6}\pi) &= -\tfrac{1}{2};\\
f^{(3)}(x) &= -\cos x, & f^{(3)}(\tfrac{1}{6}\pi) &= -\tfrac{1}{2}\sqrt{3}.
\end{aligned}$$

Hence the cubic Taylor polynomial about $\frac{1}{6}\pi$ for the sine function is

$$\begin{aligned}
p(x) &= f\left(\tfrac{1}{6}\pi\right) + f'(\tfrac{1}{6}\pi)(x - \tfrac{1}{6}\pi)\\
&\quad + \frac{f''(\frac{1}{6}\pi)}{2!}(x - \tfrac{1}{6}\pi)^2 + \frac{f^{(3)}(\frac{1}{6}\pi)}{3!}(x - \tfrac{1}{6}\pi)^3\\
&= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}(x - \tfrac{1}{6}\pi) - \tfrac{1}{4}(x - \tfrac{1}{6}\pi)^2\\
&\quad - \tfrac{1}{12}\sqrt{3}(x - \tfrac{1}{6}\pi)^3.
\end{aligned}$$

Solution 2.4

(a) You saw in the comment on Example 2.1 that the Taylor polynomial of degree n about 0 for the function $f(x) = e^x$ is

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Using this fact, and calculating values to six decimal places, we obtain

$$\begin{aligned}
p_1(-0.05) &= 1 + (-0.05) = 0.95,\\
p_2(-0.05) &= p_1(-0.05) + \frac{1}{2}(-0.05)^2\\
&= 0.951\,25,\\
p_3(-0.05) &= p_2(-0.05) + \frac{1}{6}(-0.05)^3\\
&= 0.951\,229,\\
p_4(-0.05) &= p_3(-0.05) + \frac{1}{24}(-0.05)^4\\
&= 0.951\,229.
\end{aligned}$$

The values of $p_3(-0.05)$ and $p_4(-0.05)$ agree to six decimal places, so it is likely that

$$e^{-0.05} = 0.9512,$$

to four decimal places. (This is the case.)

(b) Using formula (2.1) on page 17, we find that, for example, the Taylor polynomial of degree 8 about 0 for the cosine function is

$$p_8(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8.$$

We work with Taylor polynomials of even degree only, since each Taylor polynomial of odd degree is identical to the Taylor polynomial of degree one less.

We obtain, to eight decimal places,

$$\begin{aligned}
p_2(0.2) &= 1 - \frac{1}{2!}(0.2)^2 = 0.98,\\
p_4(0.2) &= p_2(0.2) + \frac{1}{4!}(0.2)^4 = 0.980\,066\,67,\\
p_6(0.2) &= p_4(0.2) - \frac{1}{6!}(0.2)^6 = 0.980\,066\,58,\\
p_8(0.2) &= p_6(0.2) + \frac{1}{8!}(0.2)^8 = 0.980\,066\,58.
\end{aligned}$$

The values of $p_6(0.2)$ and $p_8(0.2)$ agree to eight decimal places, so it is likely that

$$\cos(0.2) = 0.980\,067,$$

to six decimal places. (This is the case.)

Solution 3.1

(a) From the solution for Activity 2.1(a), the values of $f^{(k)}(0)$ form the repeating sequence 1, 0, -1, 0, 1, ... Hence the Taylor series for $f(x) = \cos x$ about 0 is

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots.$$

(b) Similarly, from the solution for Activity 2.1(b), the values of $f^{(k)}(0)$ form the repeating sequence 0, 1, 0, -1, 0, ... Hence the Taylor series for $f(x) = \sin x$ about 0 is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots.$$

Solution 3.2

To find the Taylor series about 1 for the function $f(x) = \sqrt{x} = x^{1/2}$, we evaluate $f(1)$, $f'(1)$, $f''(1)$, $f^{(3)}(1)$, and so on. We have:

$$f(x) = x^{1/2}, \quad f(1) = 1;$$
$$f'(x) = \tfrac{1}{2}x^{-1/2}, \quad f'(1) = \tfrac{1}{2};$$
$$f''(x) = -\tfrac{1}{2} \times \tfrac{1}{2}x^{-3/2}, \quad f''(1) = -\tfrac{1}{2} \times \tfrac{1}{2};$$
$$f^{(3)}(x) = \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}x^{-5/2}, \quad f^{(3)}(1) = \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2};$$
$$f^{(4)}(x) = -\tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}x^{-7/2},$$
$$f^{(4)}(1) = -\tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2};$$
$$f^{(5)}(x) = \tfrac{7}{2} \times \tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}x^{-9/2},$$
$$f^{(5)}(1) = \tfrac{7}{2} \times \tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}.$$

Hence the Taylor series about 1 for the function $f(x) = \sqrt{x}$ is

$$\begin{aligned}
&f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\
&\quad + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots \\
&= 1 + \tfrac{1}{2}(x-1) - \frac{1}{2^2 2!}(x-1)^2 \\
&\quad + \frac{3 \times 1}{2^3 3!}(x-1)^3 - \frac{5 \times 3 \times 1}{2^4 4!}(x-1)^4 \\
&\quad + \frac{7 \times 5 \times 3 \times 1}{2^5 5!}(x-1)^5 - \cdots .
\end{aligned}$$

The general pattern of terms is clear.

Solution 3.3

Taking $\alpha = \frac{1}{2}$ in the binomial series gives

$$\begin{aligned}
&(1+x)^{1/2} \\
&= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 \\
&\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 \\
&\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{5!}x^5 - \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1 \times 3}{2^3 3!}x^3 \\
&\quad - \frac{1 \times 3 \times 5}{2^4 4!}x^4 + \frac{1 \times 3 \times 5 \times 7}{2^5 5!}x^5 - \cdots .
\end{aligned}$$

This series is valid for $-1 < x < 1$.

(Notice the similarity between this series and the series in the solution to Activity 3.2. As you will see in Section 4, the series about 1 for $\sqrt{x}$ in Activity 3.2 can be obtained from the above series about 0 for $\sqrt{1+x}$ by replacing each occurrence of x by $x - 1$.)

Solution 3.4

The cubic Taylor polynomial about 0 for the function $f(x) = \ln(1+x)$ is obtained from the Taylor series for $\ln(1+x)$ by deleting all the terms after $\frac{1}{3}x^3$ to give

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3.$$

Solution 3.5

Using the series from the solution to Activity 3.3, we obtain, to five decimal places,

$$\begin{aligned}
p_1(0.1) &= 1 + \tfrac{1}{2}(0.1) = 1.05, \\
p_2(0.1) &= p_1(0.1) - \frac{1}{2^2 2!}(0.1)^2 = 1.048\,75, \\
p_3(0.1) &= p_2(0.1) + \frac{1 \times 3}{2^3 3!}(0.1)^3 = 1.048\,81, \\
p_4(0.1) &= p_3(0.1) - \frac{1 \times 3 \times 5}{2^4 4!}(0.1)^4 = 1.048\,81.
\end{aligned}$$

The values of $p_3(0.1)$ and $p_4(0.1)$ agree to five decimal places, so it is likely that

$$\sqrt{1.1} = 1.049,$$

to three decimal places. (This is the case.)

Solution 4.1

(a) The Taylor series about 0 for $1/(1-x)$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

for $-1 < x < 1$. Therefore the Taylor series about 0 for $1/(1+x) = 1/(1-(-x))$ is

$$\begin{aligned}
\frac{1}{1+x} &= 1 + (-x) + (-x)^2 + (-x)^3 + \cdots \\
&= 1 - x + x^2 - x^3 + \cdots .
\end{aligned}$$

This series is valid for $-1 < -x < 1$; that is, for $-1 < x < 1$.

(b) The Taylor series about 0 for $\ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots ,$$

for $-1 < x < 1$. Therefore the Taylor series about 0 for $\ln(1-x) = \ln(1+(-x))$ is

$$\begin{aligned}
&\ln(1-x) \\
&= (-x) - \tfrac{1}{2}(-x)^2 + \tfrac{1}{3}(-x)^3 - \tfrac{1}{4}(-x)^4 + \cdots \\
&= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots .
\end{aligned}$$

This series is valid for $-1 < -x < 1$; that is, for $-1 < x < 1$.

(c) The Taylor series about 0 for $\ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots,$$

for $-1 < x < 1$. Therefore the Taylor series about 0 for $\ln(1+3x)$ is

$$\begin{aligned}
&\ln(1+3x)\\
&= (3x) - \tfrac{1}{2}(3x)^2 + \tfrac{1}{3}(3x)^3 - \tfrac{1}{4}(3x)^4 + \cdots\\
&= 3x - \frac{3^2}{2}x^2 + \frac{3^3}{3}x^3 - \frac{3^4}{4}x^4 + \cdots.
\end{aligned}$$

This series is valid for $-1 < 3x < 1$; that is, for $-\frac{1}{3} < x < \frac{1}{3}$.

(d) The Taylor series about 0 for e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,$$

for $x \in \mathbb{R}$. Therefore the Taylor series about 0 for e^{x^3} is

$$\begin{aligned}
e^{x^3} &= 1 + (x^3) + \frac{1}{2!}(x^3)^2 + \frac{1}{3!}(x^3)^3 + \cdots\\
&= 1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \cdots,
\end{aligned}$$

for $x \in \mathbb{R}$.

Solution 4.2

The Taylor series about 0 for $1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

for $-1 < x < 1$. Replacing x by $x-1$ in this equation, we obtain

$$\frac{1}{1+(x-1)} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

But $1/(1+(x-1)) = 1/x$. Therefore the Taylor series about 1 for $1/x$ is

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

This series is valid for $-1 < x-1 < 1$; that is, for $0 < x < 2$.

Solution 4.3

Using the Taylor series for $\ln(1+x)$ and the series for $\ln(1-x)$ from Activity 4.1(b), we have

$$\begin{aligned}
&\ln(1+x) - \ln(1-x)\\
&= (x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots)\\
&\quad -(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \cdots)\\
&= 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \cdots;
\end{aligned}$$

that is,

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \cdots.$$

The Taylor series for $\ln(1+x)$ and $\ln(1-x)$ are each valid for $-1 < x < 1$, so the series derived here is also valid for $-1 < x < 1$.

Solution 4.4

(a) Taking $\alpha = -2$ in the binomial series gives

$$\begin{aligned}
\frac{1}{(1+x)^2} &= 1 - 2x + \frac{(-2)(-3)}{2!}x^2\\
&\quad + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots\\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots,
\end{aligned}$$

for $-1 < x < 1$.

(b) Using the result of part (a) with x replaced by $x/3$, we obtain

$$\begin{aligned}
&\frac{1}{(3+x)^2}\\
&= \frac{1}{3^2} \times \frac{1}{(1+x/3)^2}\\
&= \frac{1}{3^2}\left(1 - 2\left(\frac{x}{3}\right) + 3\left(\frac{x}{3}\right)^2 - 4\left(\frac{x}{3}\right)^3 + \cdots\right)\\
&= \frac{1}{3^2}\left(1 - \frac{2}{3}x + \frac{3}{3^2}x^2 - \frac{4}{3^3}x^3 + \cdots\right)\\
&= \frac{1}{3^2} - \frac{2}{3^3}x + \frac{3}{3^4}x^2 - \frac{4}{3^5}x^3 + \cdots.
\end{aligned}$$

This series is valid for $-1 < x/3 < 1$; that is, for $-3 < x < 3$.

Solution 4.5

We use the formula $\sinh x = \frac{1}{2}(e^x - e^{-x})$. Now

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

for $x \in \mathbb{R}$, and so

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots,$$

for $x \in \mathbb{R}$. Therefore

$$\begin{aligned}
\sinh x &= \tfrac{1}{2}(e^x - e^{-x})\\
&= \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\right.\\
&\qquad \left. - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right)\\
&= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots,
\end{aligned}$$

for $x \in \mathbb{R}$.

Solution 4.6

(a) Using the Taylor series about 0 for $\sin x$, we obtain

$$\begin{aligned} x^2 \sin x &= x^2\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) \\ &= x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 - \cdots, \end{aligned}$$

for $x \in \mathbb{R}$.

(b) Using the Taylor series about 0 for $\cos x$, we obtain

$$\begin{aligned} &(1+x)\cos x \\ &= (1+x)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\ &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\ &\qquad + x\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\ &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\ &\qquad + \left(x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \cdots\right) \\ &= 1 + x - \frac{1}{2!}x^2 - \frac{1}{2!}x^3 + \frac{1}{4!}x^4 + \frac{1}{4!}x^5 - \cdots, \end{aligned}$$

for $x \in \mathbb{R}$.

Solution 4.7

Using the Taylor series about 0 for $1/(1+x)$ and $\sin x$, we obtain

$$\begin{aligned} &\frac{\sin x}{1+x} \\ &= (1 - x + x^2 - x^3 + \cdots)\left(x - \frac{1}{3!}x^3 + \cdots\right) \\ &= \left(x - \frac{1}{3!}x^3 + \cdots\right) - x(x - \cdots) + x^2(x - \cdots) - \cdots \\ &= (x - \tfrac{1}{6}x^3 + \cdots) - (x^2 - \cdots) + (x^3 - \cdots) - \cdots \\ &= x - x^2 + \tfrac{5}{6}x^3 - \cdots. \end{aligned}$$

Hence the cubic Taylor polynomial about 0 for $f(x) = (\sin x)/(1+x)$ is

$$p_3(x) = x - x^2 + \tfrac{5}{6}x^3.$$

(In the above working all terms which could eventually lead to terms of power 3 or less were retained, and any terms which affect only terms of power 4 or more were ignored.)

Solution 4.8

The Taylor series about 0 for e^x is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots.$$

Differentiating this series gives

$$\begin{aligned} &0 + 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \cdots \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \end{aligned}$$

which is the same series, as required.

Solution 4.9

Integrating both sides of the given equation yields

$$\arctan x = c + x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots,$$

for $-1 < x < 1$, where c is an arbitrary constant. Taking $x = 0$ gives $\arctan 0 = c$; hence $c = 0$. Therefore

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots,$$

for $-1 < x < 1$.

Solution 4.10

(a) Taking $\alpha = -\frac{1}{2}$ in the binomial series gives

$$\begin{aligned} \frac{1}{\sqrt{1+x}} &= 1 - \tfrac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \cdots \\ &= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots, \end{aligned}$$

for $-1 < x < 1$.

(b) Replacing x by $-x^2$ in the above series gives

$$\begin{aligned} \frac{1}{\sqrt{1-x^2}} &= 1 - \tfrac{1}{2}(-x^2) + \tfrac{3}{8}(-x^2)^2 + \cdots \\ &= 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \cdots. \end{aligned}$$

This Taylor series is valid for $-1 < -x^2 < 1$. The left-hand inequality here is $-1 < -x^2$, which is equivalent to $1 > x^2$; that is, $-1 < x < 1$. The right-hand inequality is $-x^2 < 1$, which is equivalent to $x^2 > -1$ and therefore does not place any restriction on x, since the square of any real number is non-negative. Thus the Taylor series is valid for $-1 < x < 1$.

(c) Integrating both sides of the above equation gives

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \left(1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \cdots\right) dx;$$

that is,

$$\arcsin x = c + x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \cdots,$$

for $-1 < x < 1$, where c is an arbitrary constant. Taking $x = 0$ gives $\arcsin 0 = c$; hence $c = 0$. Therefore

$$\arcsin x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \cdots,$$

for $-1 < x < 1$.

Solutions to Exercises

Solution 1.1

Here $f(x) = 1/x^2$, so $(1, f(1)) = (1, 1/1^2) = (1, 1)$. Also $f'(x) = -2/x^3$, so the gradient of the curve at $(1,1)$ is $f'(1) = -2/1^3 = -2$. Thus the required line passes through the point $(1,1)$ and has gradient -2.

Since its gradient is -2, the line has an equation of the form $y = a_0 - 2x$. Since it passes through $(1,1)$, we have $1 = a_0 - 2 \times 1$; that is, $a_0 = 3$. Thus the line has equation $y = 3 - 2x$, and so the linear Taylor polynomial is $p(x) = 3 - 2x$.

Solution 1.2

Here $f(x) = (8+x)^{1/3}$, so $(0, f(0)) = (0, 8^{1/3}) = (0, 2)$. Also $f'(x) = \frac{1}{3}(8+x)^{-2/3}$, so the gradient of the curve at $(0,2)$ is $f'(0) = \frac{1}{3} \times 8^{-2/3} = \frac{1}{12}$. Thus the required line passes through the point $(0,2)$ and has gradient $\frac{1}{12}$.

Since its gradient is $\frac{1}{12}$, the line has an equation of the form $y = a_0 + \frac{1}{12}x$. Since it passes through $(0,2)$, we have $2 = a_0 + \frac{1}{12} \times 0$; that is, $a_0 = 2$. Thus the line has equation $y = 2 + \frac{1}{12}x$, and so the linear Taylor polynomial is $p(x) = 2 + \frac{1}{12}x$.

The approximation for $\sqrt[3]{8.01} = f(0.01)$ given by the polynomial p is

$$\begin{aligned} p(0.01) &= 2 + \tfrac{1}{12} \times 0.01 \\ &= 2.000\,833, \end{aligned}$$

to six decimal places. The remainder is

$$\begin{aligned} r(0.01) &= f(0.01) - p(0.01) \\ &= -0.000\,000, \end{aligned}$$

to six decimal places. (To seven decimal places, $r(0.01) = -0.000\,0003$.)

Solution 1.3

Let the polynomial that we seek be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

First we ensure that $p(0) = f(0)$. We have

$$\begin{aligned} f(x) &= \tan x, & p(x) &= a_0 + a_1 x + a_2 x^2, \\ f(0) &= \tan 0 = 0, & p(0) &= a_0. \end{aligned}$$

Thus we must have $a_0 = 0$.

Next we ensure that $p'(0) = f'(0)$. We have

$$\begin{aligned} f'(x) &= \sec^2 x, & p'(x) &= a_1 + 2a_2 x, \\ f'(0) &= \sec^2 0 = 1, & p'(0) &= a_1. \end{aligned}$$

Thus we must have $a_1 = 1$.

Finally we ensure that $p''(0) = f''(0)$. We have

$$\begin{aligned} f''(x) &= 2\sec x(\sec x \tan x), & p''(x) &= 2a_2, \\ f''(0) &= 2\sec 0(\sec 0 \tan 0) = 0, & p''(0) &= 2a_2. \end{aligned}$$

Thus we must have $2a_2 = 0$; that is, $a_2 = 0$.

Hence the quadratic Taylor polynomial about 0 for the tangent function is

$$p(x) = x.$$

(Notice that this is the same as the quadratic Taylor polynomial about 0 for the sine function, and that it is another example of a quadratic Taylor polynomial that is not a quadratic polynomial.)

The polynomial p gives the approximation

$$\tan(-0.1) \simeq -0.1.$$

The remainder is

$$\begin{aligned} r(-0.1) &= \tan(-0.1) - p(-0.1) \\ &= -0.100\,335 - (-0.1) \\ &= -0.000\,335, \end{aligned}$$

to six decimal places.

Solution 2.1

In each case, to find the Taylor polynomial of degree 5 for $f(x)$ about a, we evaluate $f(a)$, $f'(a)$, $f''(a)$, $f^{(3)}(a)$, $f^{(4)}(a)$ and $f^{(5)}(a)$.

(a) The evaluations for $f(x) = x^6$ and $a = 1$ are as follows:

$$\begin{aligned} f(x) &= x^6, & f(1) &= 1; \\ f'(x) &= 6x^5, & f'(1) &= 6; \\ f''(x) &= 30x^4, & f''(1) &= 30; \\ f^{(3)}(x) &= 120x^3, & f^{(3)}(1) &= 120; \\ f^{(4)}(x) &= 360x^2, & f^{(4)}(1) &= 360; \\ f^{(5)}(x) &= 720x, & f^{(5)}(1) &= 720. \end{aligned}$$

Hence the quintic Taylor polynomial about 1 for $f(x) = x^6$ is

$$\begin{aligned} p(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &\quad + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &\quad + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\ &= 1 + 6(x-1) + 15(x-1)^2 + 20(x-1)^3 \\ &\quad + 15(x-1)^4 + 6(x-1)^5. \end{aligned}$$

(b) The evaluations for $f(x) = \ln(1-x)$ and $a = 0$ are as follows:

$$\begin{aligned}
f(x) &= \ln(1-x), & f(0) &= 0;\\
f'(x) &= -(1-x)^{-1}, & f'(0) &= -1;\\
f''(x) &= -(1-x)^{-2}, & f''(0) &= -1;\\
f^{(3)}(x) &= -2(1-x)^{-3}, & f^{(3)}(0) &= -2!;\\
f^{(4)}(x) &= -3\times 2(1-x)^{-4}, & f^{(4)}(0) &= -3!;\\
f^{(5)}(x) &= -4\times 3\times 2(1-x)^{-5}, & f^{(5)}(0) &= -4!.
\end{aligned}$$

(Factorial notation has been used here to highlight the pattern in the values.)

Hence the quintic Taylor polynomial about 0 for $f(x) = \ln(1-x)$ is

$$\begin{aligned}
p(x) &= -x - \frac{1}{2!}x^2 - \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 - \frac{4!}{5!}x^5\\
&= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5.
\end{aligned}$$

(Notice the pattern in the terms; you should be able to see from the working above that this pattern continues as the degree of the Taylor polynomial increases.)

(c) The evaluations for $f(x) = \sin x$ and $a = \frac{1}{4}\pi$ are as follows:

$$\begin{aligned}
f(x) &= \sin x, & f(\tfrac{1}{4}\pi) &= \tfrac{1}{2}\sqrt{2};\\
f'(x) &= \cos x, & f'(\tfrac{1}{4}\pi) &= \tfrac{1}{2}\sqrt{2};\\
f''(x) &= -\sin x, & f''(\tfrac{1}{4}\pi) &= -\tfrac{1}{2}\sqrt{2};\\
f^{(3)}(x) &= -\cos x, & f^{(3)}(\tfrac{1}{4}\pi) &= -\tfrac{1}{2}\sqrt{2};\\
f^{(4)}(x) &= \sin x, & f^{(4)}(\tfrac{1}{4}\pi) &= \tfrac{1}{2}\sqrt{2};\\
f^{(5)}(x) &= \cos x, & f^{(5)}(\tfrac{1}{4}\pi) &= \tfrac{1}{2}\sqrt{2}.
\end{aligned}$$

Hence the quintic Taylor polynomial about $\frac{1}{4}\pi$ for $f(x) = \sin x$ is

$$\begin{aligned}
p(x) &= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\,(x - \tfrac{1}{4}\pi) - \frac{\frac{1}{2}\sqrt{2}}{2!}\,(x - \tfrac{1}{4}\pi)^2\\
&\quad - \frac{\frac{1}{2}\sqrt{2}}{3!}\,(x - \tfrac{1}{4}\pi)^3 + \frac{\frac{1}{2}\sqrt{2}}{4!}\,(x - \tfrac{1}{4}\pi)^4\\
&\quad + \frac{\frac{1}{2}\sqrt{2}}{5!}\,(x - \tfrac{1}{4}\pi)^5\\
&= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\,(x - \tfrac{1}{4}\pi) - \tfrac{1}{4}\sqrt{2}\,(x - \tfrac{1}{4}\pi)^2\\
&\quad - \tfrac{1}{12}\sqrt{2}\,(x - \tfrac{1}{4}\pi)^3 + \tfrac{1}{48}\sqrt{2}\,(x - \tfrac{1}{4}\pi)^4\\
&\quad + \tfrac{1}{240}\sqrt{2}\,(x - \tfrac{1}{4}\pi)^5.
\end{aligned}$$

(d) The evaluations for $f(x) = (1+x)^{1/2}$ and $a = 0$ are as follows:

$$\begin{aligned}
f(x) &= (1+x)^{1/2}, & f(0) &= 1;\\
f'(x) &= \tfrac{1}{2}(1+x)^{-1/2}, & f'(0) &= \tfrac{1}{2};\\
f''(x) &= -\tfrac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\tfrac{1}{4};\\
f^{(3)}(x) &= \tfrac{3}{8}(1+x)^{-5/2}, & f^{(3)}(0) &= \tfrac{3}{8};\\
f^{(4)}(x) &= -\tfrac{15}{16}(1+x)^{-7/2}, & f^{(4)}(0) &= -\tfrac{15}{16};\\
f^{(5)}(x) &= \tfrac{105}{32}(1+x)^{-9/2}, & f^{(5)}(0) &= \tfrac{105}{32}.
\end{aligned}$$

Hence the quintic Taylor polynomial about 0 for $f(x) = (1+x)^{1/2}$ is

$$\begin{aligned}
p(x) &= 1 + \tfrac{1}{2}x + \frac{(-\frac{1}{4})}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3\\
&\quad + \frac{(-\frac{15}{16})}{4!}x^4 + \frac{\frac{105}{32}}{5!}x^5\\
&= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5.
\end{aligned}$$

Solution 2.2

With 10 expressed as $9(1+\frac{1}{9})$, we have

$$\sqrt{10} = \sqrt{9(1+\tfrac{1}{9})} = 3(1+\tfrac{1}{9})^{1/2}.$$

Thus if $f(x) = (1+x)^{1/2}$, then $\sqrt{10} = 3f(\frac{1}{9})$.

Using the result of Exercise 2.1(d), we obtain, to five decimal places, the following approximations for $3f(\frac{1}{9})$:

$$\begin{aligned}
3p_1(\tfrac{1}{9}) &= 3(1 + \tfrac{1}{2}(\tfrac{1}{9})) = 3.166\,67,\\
3p_2(\tfrac{1}{9}) &= 3p_1(\tfrac{1}{9}) - 3\times\tfrac{1}{8}(\tfrac{1}{9})^2 = 3.162\,04,\\
3p_3(\tfrac{1}{9}) &= 3p_2(\tfrac{1}{9}) + 3\times\tfrac{1}{16}(\tfrac{1}{9})^3 = 3.162\,29,\\
3p_4(\tfrac{1}{9}) &= 3p_3(\tfrac{1}{9}) - 3\times\tfrac{5}{128}(\tfrac{1}{9})^4 = 3.162\,28,\\
3p_5(\tfrac{1}{9}) &= 3p_4(\tfrac{1}{9}) + 3\times\tfrac{7}{256}(\tfrac{1}{9})^5 = 3.162\,28.
\end{aligned}$$

Since $3p_4(\frac{1}{9})$ and $3p_5(\frac{1}{9})$ agree to five decimal places, it is likely that

$$\sqrt{10} = 3f(\tfrac{1}{9}) = 3.162,$$

to three decimal places. (This is the case.)

Solution 3.1

To find the Taylor series about 2 for the function $f(x) = x^{-1}$, we evaluate $f(2)$, $f'(2)$, $f''(2)$, $f^{(3)}(2)$, $f^{(4)}(2)$, and so on. We obtain

$$\begin{aligned}
f(x) &= x^{-1}, & f(2) &= \frac{1}{2};\\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2};\\
f''(x) &= 2x^{-3}, & f''(2) &= \frac{2!}{2^3};\\
f^{(3)}(x) &= -3\times 2x^{-4}, & f^{(3)}(2) &= -\frac{3!}{2^4};\\
f^{(4)}(x) &= 4\times 3\times 2x^{-5}, & f^{(4)}(2) &= \frac{4!}{2^5}.
\end{aligned}$$

The general pattern in these values should be clear.

The Taylor series about 2 for $f(x) = x^{-1}$ is

$$\begin{aligned}
&f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2\\
&\quad + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \cdots\\
&= \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 - \frac{1}{2^4}(x-2)^3\\
&\quad + \frac{1}{2^5}(x-2)^4 - \cdots.
\end{aligned}$$

Solution 3.2

Taking $\alpha = 5$ in the binomial series gives

$$\begin{aligned}(1+x)^5 &= 1 + 5x + \frac{5\times 4}{2!}x^2 + \frac{5\times 4\times 3}{3!}x^3 \\ &\quad + \frac{5\times 4\times 3\times 2}{4!}x^4 \\ &\quad + \frac{5\times 4\times 3\times 2\times 1}{5!}x^5 \\ &\quad + 0 + 0 + \cdots \\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 .\end{aligned}$$

This series is valid for $x \in \mathbb{R}$.

(It is the binomial expansion of $(1+x)^5$.)

Solution 3.3

Taking $\alpha = -\frac{1}{3}$ in the binomial series gives

$$\begin{aligned}\frac{1}{\sqrt[3]{1+x}} &= 1 + (-\tfrac{1}{3})x + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}x^2 \\ &\quad + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}x^3 + \cdots \\ &= 1 - \tfrac{1}{3}x + \tfrac{2}{9}x^2 - \tfrac{14}{81}x^3 + \cdots .\end{aligned}$$

Therefore the required cubic Taylor polynomial is

$$1 - \tfrac{1}{3}x + \tfrac{2}{9}x^2 - \tfrac{14}{81}x^3 .$$

Solution 3.4

We use the method of Subsection 2.3, obtaining the required Taylor polynomials by truncating the Taylor series about 0 for $\ln(1+x)$ at the appropriate term. We obtain, to five decimal places,

$$\begin{aligned}&p_1(0.25) = 0.25,\\ &p_2(0.25) = p_1(0.25) - \tfrac{1}{2}(0.25)^2 = 0.218\,75,\\ &p_3(0.25) = p_2(0.25) + \tfrac{1}{3}(0.25)^3 = 0.223\,96,\\ &p_4(0.25) = p_3(0.25) - \tfrac{1}{4}(0.25)^4 = 0.222\,98,\\ &p_5(0.25) = p_4(0.25) + \tfrac{1}{5}(0.25)^5 = 0.223\,18,\\ &p_6(0.25) = p_5(0.25) - \tfrac{1}{6}(0.25)^6 = 0.223\,14,\\ &p_7(0.25) = p_6(0.25) + \tfrac{1}{7}(0.25)^7 = 0.223\,15,\\ &p_8(0.25) = p_7(0.25) - \tfrac{1}{8}(0.25)^8 = 0.223\,14,\\ &p_9(0.25) = p_8(0.25) + \tfrac{1}{9}(0.25)^9 = 0.223\,14.\end{aligned}$$

Since $p_8(0.25)$ and $p_9(0.25)$ agree to five decimal places, it is likely that

$$\ln(1.25) = 0.223$$

to three decimal places. (This is the case.)

Solution 4.1

The Taylor series about 0 for the function $1/(1-x)$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ,$$

for $-1 < x < 1$. Therefore the Taylor series about 0 for the function $1/(1-4x^2)$ is

$$\begin{aligned}\frac{1}{1-4x^2} &= 1 + (4x^2) + (4x^2)^2 + (4x^2)^3 + \cdots \\ &= 1 + 4x^2 + 16x^4 + 64x^6 + \cdots .\end{aligned}$$

The Taylor series for $1/(1-x)$ is valid for $-1 < x < 1$, so the series for $1/(1-4x^2)$ is valid for $-1 < 4x^2 < 1$. The left-hand inequality here is $-1 < 4x^2$, which does not place any restriction on x, since the square of any real number is non-negative. The right-hand inequality is $4x^2 < 1$, which is equivalent to $x^2 < \frac{1}{4}$; that is, $-\frac{1}{2} < x < \frac{1}{2}$. Thus the Taylor series for $1/(1-4x^2)$ is valid for $-\frac{1}{2} < x < \frac{1}{2}$.

Solution 4.2

The Taylor series about 0 for the function $\ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots ,$$

for $-1 < x < 1$. Therefore the Taylor series about 2 for the function $\ln(x-1) = \ln(1+(x-2))$ is

$$\begin{aligned}\ln(x-1) &= (x-2) - \tfrac{1}{2}(x-2)^2 + \tfrac{1}{3}(x-2)^3 \\ &\quad - \tfrac{1}{4}(x-2)^4 + \cdots .\end{aligned}$$

The Taylor series about 0 for $\ln(1+x)$ is valid for $-1 < x < 1$, so the above series about 2 for $\ln(x-1)$ is valid for $-1 < x-2 < 1$; that is, for $1 < x < 3$.

Solution 4.3

The Taylor series about 0 for $\sin x$ is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Therefore the Taylor series about 0 for $\sin(-x)$ is

$$\begin{aligned}\sin(-x) &= (-x) - \frac{1}{3!}(-x)^3 + \frac{1}{5!}(-x)^5 - \cdots \\ &= -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots .\end{aligned}$$

Using this series, and the Taylor series for e^x, we obtain:

$$\begin{aligned}&e^x \sin(-x) \\ &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ &\quad \times \left(-x + \frac{1}{3!}x^3 + \cdots\right) \\ &= \left(-x + \tfrac{1}{6}x^3 + \cdots\right) \\ &\quad + x\left(-x + \tfrac{1}{6}x^3 + \cdots\right) \\ &\quad + \tfrac{1}{2}x^2(-x + \cdots) + \tfrac{1}{6}x^3(-x + \cdots) + \cdots \\ &= -x - x^2 - \tfrac{1}{3}x^3 + 0x^4 + \cdots .\end{aligned}$$

Thus the required quartic Taylor polynomial is

$$-x - x^2 - \tfrac{1}{3}x^3.$$

(In the above calculation at each stage only those terms that could result in terms of power 4 or less were retained. Notice that this is an example of a Taylor polynomial of degree 4 whose polynomial degree is less than 4.)

Solution 4.4

(a) Rearranging the equation $1.25 = (1 + x)/(1 - x)$ gives

$$1.25(1 - x) = 1 + x;$$

that is

$$1.25 - 1.25x = 1 + x.$$

Hence $2.25x = 0.25$, so

$$x = \tfrac{1}{9}\,.$$

(Alternatively, this value of x can be obtained by using the equation in the margin opposite the comment on Activity 4.3.)

Let

$$f(x) = \ln\left(\frac{1+x}{1-x}\right);$$

then $\ln(1.25) = f(\frac{1}{9})$. In Activity 4.3 we found that

$$f(x) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \cdots,$$

for $-1 < x < 1$. Using the Taylor polynomials obtained from this series, we obtain, to five decimal places,

$$p_1(\tfrac{1}{9}) = 2 \times \tfrac{1}{9} = 0.222\,22,$$

$$p_3(\tfrac{1}{9}) = p_1(\tfrac{1}{9}) + \tfrac{2}{3}(\tfrac{1}{9})^3 = 0.223\,14,$$

$$p_5(\tfrac{1}{9}) = p_3(\tfrac{1}{9}) + \tfrac{2}{5}(\tfrac{1}{9})^5 = 0.223\,14.$$

Since $p_3(\frac{1}{9})$ and $p_5(\frac{1}{9})$ agree to five decimal places, it is likely that

$$\ln(1.25) = f(\tfrac{1}{9}) = 0.223,$$

to three decimal places. (This is the case.)

(b) This method of calculating an approximate value for $\ln(1.25)$ is preferable to that of Exercise 3.4; it involves the evaluation of only three terms, whereas the earlier method involved evaluating nine terms.

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ISBN 0 7492 5561 7